

FUTURE COMPLETE VACUUM SPACETIMES

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ABSTRACT. In this paper we prove a global existence theorem, in the direction of cosmological expansion, for sufficiently small perturbations of a family of spatially compact variants of the $k = -1$ Friedmann–Robertson–Walker vacuum spacetime. We use a special gauge defined by constant mean curvature slicing and a spatial harmonic coordinate condition, and develop energy estimates through the use of the Bel–Robinson energy and its higher order generalizations. In addition to the smallness condition on the data, we need a topological constraint on the spatial manifold to exclude the possibility of a non-trivial moduli space of flat spacetime perturbations, since the latter could not be controlled by curvature-based energies such as those of Bel–Robinson type. Our results also demonstrate causal geodesic completeness of the perturbed spacetimes (in the expanding direction) and establish precise rates of decay towards the background solution which serves as an attractor asymptotically.

1. INTRODUCTION

In this paper we establish global existence and asymptotic behavior, in the cosmologically expanding direction, for a family of spatially compact, vacuum solutions to the 3+1 dimensional Einstein equations for sufficiently small perturbations of certain known “background” solutions. The backgrounds we consider are the spatially compactified variants of the familiar vacuum $k = -1$ Friedmann–Robertson–Walker (FRW) solution, which exist on any 4-manifold \bar{M} of the form $(0, \infty) \times M$, where M is a compact hyperbolic 3-manifold (i.e., a manifold admitting a Riemannian metric with constant negative sectional curvature).

Let γ be the standard hyperbolic metric with sectional curvature -1 on M . Then $(\bar{M}, \bar{\gamma})$ given by

$$\begin{aligned}\bar{M} &= (0, \infty) \times M \\ \bar{\gamma} &= -d\rho \otimes d\rho + \rho^2 \gamma\end{aligned}$$

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is a flat spacetime, locally isometric to the $k = -1$ vacuum FRW model, which we shall call a hyperbolic cone spacetime. It has a big bang singularity as $\rho \searrow 0$ but expands to infinite volume as $\rho \nearrow \infty$. The vector field $\rho \frac{\partial}{\partial \rho}$ is a timelike homothetic Killing field on $(\bar{M}, \bar{\gamma})$ so that these backgrounds are continuously self-similar. We shall be considering sufficiently small perturbations of such hyperbolic cone spacetimes to the future of an arbitrary $\rho = \text{constant}$ Cauchy surface under the additional topological restriction that $(\bar{M}, \bar{\gamma})$ be “rigid” in a sense that we shall define more fully below. The rigidity assumption serves to eliminate the possibility of making non-trivial but still flat perturbations of the chosen backgrounds.

Our main result treats the vacuum Einstein equations on \bar{M} and proves global existence in the expanding direction for initial data sufficiently close to data for $(\bar{M}, \bar{\gamma})$. More precisely, we show that the maximal globally hyperbolic future vacuum development (\bar{M}, \bar{g}) of such data is causally geodesically complete and globally foliated by constant mean curvature (CMC) hypersurfaces in the expanding direction. We further show that the metric \bar{g} decays asymptotically to $\bar{\gamma}$ at a well-defined rate (that correctly predicted by linearized theory) and give the sharp rate of decay. In this sense our result may be viewed as a nonlinear stability result for the future evolution. We could also view it as implying nonlinear instability for the past evolution but, since our arguments are insufficient to treat global evolution in the past direction, we shall concentrate here on the expanding direction. Since the formation of black holes would be expected to violate geodesic completeness towards the future, we can also interpret our smallness condition in the data as sufficient to exclude the formation of black holes.

We work in a specific gauge defined by constant mean curvature slicing and a spatial harmonic coordinate condition which serves to kill off certain second order terms in the spatial Ricci tensor, reducing it to a nonlinear elliptic operator on the metric. This in turn effectively reduces the evolution equations for the spatial metric to nonlinear wave equations wherein, however, the lapse function and shift vector field are determined by an associated set of (linear) elliptic equations. Local existence and well-posedness for the Einstein equations in this gauge was established in [1] along with a continuation principle which provides the needed criterion for proving global existence.

The main tool we employ for our global existence proof is an energy argument based on the Bel–Robinson energy and its higher order generalization, which we define. The Bel–Robinson energy for a vacuum spacetime is basically an L^2 -norm of spacetime curvature on a given Cauchy hypersurface, and its higher order generalization incorporates the L^2 -norm of the spatial gradient of this same curvature. One of the key steps in our proof will be to show that, in our chosen gauge, this generalized Bel–Robinson energy bounds an $H^3 \times H^2$ norm of the perturbed first and second fundamental forms of a CMC slice in the spacetime (\bar{M}, \bar{g}) .

Nontrivial spacetime perturbations which preserve flatness are invisible to such purely curvature based energies, and this is the reason we have been forced to impose an additional rigidity condition upon the hyperbolic manifolds that we consider. Already by Mostow rigidity one cannot perturb the flat metric $\bar{\gamma} = -d\rho \otimes d\rho + \rho^2 \gamma$ to another flat one by simply deforming the hyperbolic metric γ on M , but there can be more subtle ways of deforming $\bar{\gamma}$ on \bar{M} that preserve flatness. These arise whenever (M, γ) admits so-called nontrivial traceless Codazzi tensors. Our rigidity requirement is that (M, γ) be such as to exclude such tensors—a condition which is known to be satisfied for a non-empty set of hyperbolic manifolds.

The Bel–Robinson energy is of course not a conserved quantity but, together with its higher order generalization, can actually be shown to decay in the expanding direction for sufficiently small perturbations of a hyperbolic cone spacetime. The main source of this decay is the overall expansion of the universe which leads to an omnipresent term of good sign, proportional to the energy itself, in the time derivative of this energy. A corresponding result holds for the generalized energy. The remaining terms in the time derivative in general have no clear sign but fortunately can be bounded by a power greater than unity of the generalized energy itself. When the initial value of the generalized energy is sufficiently small this implies decay to the future at an asymptotically well-defined rate and leads to our main result.

While we shall not pursue this issue here, there seems to be a straightforward way to remove the rigidity constraint and thereby deal with arbitrary hyperbolic M . This involves supplementing the Bel–Robinson energies considered here by another non-curvature-based energy called the reduced Hamiltonian. As discussed in [7] this quantity is always monotonically decaying towards the future (even for large data) but bounds at most the rather weak $H^1 \times L^2$ norm of the CMC Cauchy data. However this should more than suffice to control the finite dimensional space of moduli parameters which arises in the case of non-rigid M but is invisible to the Bel–Robinson energies.

Apart from general Lorentzian geometry results such as singularity theorems and conclusions drawn from the study of explicit solutions, very little is known about the global properties of generic 3+1 dimensional Einstein spacetimes, with or without matter, and present PDE technology is far from being applicable to the study of such global questions, except in the case of small data.

In [5] Christodoulou and Klainerman proved the nonlinear stability of 3+1 dimensional Minkowski space, i.e., a small data global existence result together with precise statements about the asymptotic decay of the metric to the Minkowski metric. This proof was based on a bootstrap argument using decay estimates for suitably defined Bel–Robinson energies. A central element in the proof was the construction of approximate Killing and conformal Killing fields, which were then used in a way which is analogous to the way in which true Killing and conformal Killing fields of Minkowski space

are used in the proof of the Klainerman Sobolev inequalities for solutions of the wave equation on Minkowski space.

In still earlier work [9] Friedrich had proven global existence to the future of a Cauchy surface for the development of data sufficiently close to that of a hyperboloid in Minkowski space, with asymptotic behavior compatible with a regular conformal compactification in the sense of Penrose. This result used the fact that the conformal compactification of such spacetimes has a regular null boundary (Scri) and exploited Cauchy stability for a conformally regular first order symmetric hyperbolic system of field equations deduced from the Einstein equations. Roughly speaking, local existence for the conformally regular system can correspond, for sufficiently small data, to global existence for the conformally related, physical spacetime.

Our argument is close in spirit to that of Christodoulou and Klainerman but is much simpler than theirs by virtue of the universal energy decay described above. The source of this decay can easily be seen in linear perturbation theory by exploiting the fact that $\rho \frac{\partial}{\partial \rho}$ is a timelike homothetic Killing field in the background. One readily constructs from this an exactly conserved quantity for the linearized equations which differs from the (linearized analogue of the) Bel–Robinson energy we consider by a multiplicative factor in the time variable ρ . This gives immediately the specific decay rate predicted by linearized theory and our arguments ultimately show that this is the precise decay rate asymptotically realized by solutions to the (small data) nonlinear problem.

Our arguments are also similar in spirit to those of [3] in which Choquet–Bruhat and Moncrief treat perturbations of certain $\mathcal{U}(1)$ –symmetric vacuum spacetimes on $\mathbb{R} \times \Sigma \times S^1$, where Σ is a higher genus surface and in which the $\mathcal{U}(1)$ (Killing) symmetry is imposed along the circular fibers of the product bundle $\mathbb{R} \times \Sigma \times S^1 \rightarrow \mathbb{R} \times \Sigma$. Their results also use energy arguments which exploit the universal expansion to obtain decay for small data. A significant generalization of that work is presented in the article by Choquet–Bruhat in the present volume, wherein she removes the restriction to “polarized” solutions adopted in the earlier work. For the case of linearized perturbations, Fischer and Moncrief [8] have analyzed the stability of higher dimensional analogues of the hyperbolic cone spacetimes described above wherein the hyperbolic metric γ is replaced by an arbitrary Einstein metric with negative Einstein constant. These of course include the higher dimensional hyperbolic metrics but in fact comprise a much larger set. It now seems likely that the nonlinear stability problem for these spacetimes can be handled by a combination of the methods employed herein and in the article by Choquet–Bruhat.

We now give a more precise description of our main results. Let g be a Riemannian metric on M and let k be a symmetric covariant 2–tensor on M . We call (M, g, k) a vacuum data set for the Einstein equations if (g, k) satisfy the vacuum constraint equations, reviewed in section 2.1 below. Given such a vacuum data set there is a unique maximal Cauchy development (\bar{M}, \bar{g})

of (M, g, k) which contains the latter as an embedded Cauchy hypersurface. Our results concern the structure of (\bar{M}, \bar{g}) , especially to the future of the Cauchy hypersurface, for (g, k) sufficiently close to the data corresponding to a rigid hyperbolic cone spacetime $(\bar{M}, \bar{\gamma})$. We show in this case that, in the expanding direction, (\bar{M}, \bar{g}) is globally foliated by hypersurfaces of constant mean curvature and that (\bar{M}, \bar{g}) is causally geodesically complete in this (future) direction. In particular, (\bar{M}, \bar{g}) is inextendible in the expanding direction and thus our results support the strong cosmic censorship hypothesis.

Our main result is summarized as follows.

Theorem 1.1. *Let (M, γ) be a compact hyperbolic 3-manifold and assume that (M, γ) is rigid (i.e., admits no nontrivial traceless Codazzi tensors). Assume that (M, g^0, k^0) is a CMC vacuum data set with $(g^0, k^0) \in H^s \times H^{s-1}$, $s \geq 3$, having $t_0 = \text{tr}_{g^0} k^0 = \text{constant} < 0$. Then there is an $\epsilon > 0$ so that if*

$$\| \frac{t_0^2}{9} g^0 - \gamma \|_{H^3} + \| \frac{t_0}{3} k^0 - \gamma \|_{H^2} < \epsilon$$

then

- (1) *The maximal Cauchy development (\bar{M}, \bar{g}) of the vacuum data set (M, g^0, k^0) has a global CMC foliation in the expanding direction (to the future of M_{t_0} in CMC time $t = \text{tr}_g k$).*
- (2) *(\bar{M}, \bar{g}) is future causally geodesically complete.*

Remark 1.1. (1) *Under our conventions, c.f. section 2, the standard hyperboloid $\{\langle x, x \rangle = -1\}$ in $I^+(\{0\}) \subset \mathbb{R}^{3,1}$ has mean curvature -3 and $\text{Vol}(M, g)$ increases as $t \nearrow 0$.*

(2) *$(\frac{t^2}{9}g, \frac{|t|}{3}k)$ are rescaled Cauchy data that reduce to $(\gamma, -\gamma)$ for the background solution. Our energy arguments show that the rescaled data approach their background values at a well-defined asymptotic rate as $t = \text{tr}_g k \nearrow 0$.*

(3) *By exploiting the scaling with respect to t at t_0 one can satisfy the smallness condition for initial data (g^0, k^0) corresponding to arbitrarily large initial spacetime curvature. In this sense one can choose the initial hypersurface to be “close to the singularity”.*

In outline our paper proceeds as follows. Some preliminaries and a discussion of the Einstein equations in our chosen gauge including a review of the local existence theorem proven in [1], are given in sections 2 and 2.1. Sections 2.2–2.4 discuss the background spacetimes, the constraint set for the perturbed spacetimes and the rigidity condition needed to exclude the occurrence of a moduli space of flat perturbations. Section 3 introduces Weyl fields in the spirit of Christodoulou and Klainerman and presents the field equations they satisfy when Einstein’s equations are imposed. Section 4 discusses the Bel–Robinson energy and its higher order generalization and

computes the time derivative of these quantities in the chosen gauge. Section 4.1 describes the scale-free variants of these energies that are used in our estimates and section 4.2 gives the calculation which shows how these energies actually bound Sobolev norms of the perturbed data in the rigid case. Sections 5 and 5.1 discuss estimates and the differential inequalities satisfied by our rescaled Bel–Robinson energies. The global existence proof is completed in section 6 and section 6.1 establishes causal geodesic completeness. A number of useful definitions and identities are collected in the appendix.

2. PRELIMINARIES

Let \bar{M} be a spacetime, i.e an $n+1$ dimensional manifold with Lorentz metric \bar{g} of signature $-+\cdots+$ and covariant derivative $\bar{\nabla}$. We denote by $\langle \cdot, \cdot \rangle$ the scalar product defined by \bar{g} on $T\bar{M}$. Let $M \subset \bar{M}$ be a spacelike hypersurface with timelike normal T , $\langle T, T \rangle = -1$, and let t be a time function on a neighborhood of M . Then we can introduce local coordinates $(t, x^i, i = 1, \dots, n)$ on \bar{M} so that x^i are coordinates on the level sets M_t of t . We will often drop the subscript t on M_t and associated fields.

Let $\partial_t = \partial/\partial t$ be the coordinate vector field corresponding to t . The lapse function N and shift vectorfield X of the foliation $\{M_t\}$ are defined by $\partial_t = NT + X$. Assume T is future directed so that $N > 0$. The space–time metric \bar{g} takes the form

$$\bar{g} = -N^2 dt \otimes dt + g_{ij}(dx^i + X^i dt) \otimes (dx^j + X^j dt). \quad (2.1)$$

Let $\{e_i\}_{i=1,\dots,n}$ be a Fermi-propagated orthonormal frame tangent to M_t , i.e. $\langle \bar{\nabla}_T e_i, e_j \rangle = 0$, $\forall i, j$, with dual frame $\{e^i\}_{i=1}^n$. If one drops the assumption that the frame is Fermi propagated, then in general $\bar{\nabla}_T e_i = \bar{\nabla}_T^{\parallel} e_i + (N^{-1} \nabla_i N) T$, where $\bar{\nabla}_T^{\parallel} e_i$ denotes the tangential part of $\bar{\nabla}_T e_i$. With $e_0 = T$, $\{e_\mu\}_{\mu=0}^n$ is an ON frame on \bar{M} , adapted to the foliation $\{M_t\}$. We will use the convention that lower case latin indices run over $1, \dots, n$, while greek indices run over $0, \dots, n$. Our conventions for curvature as well as some useful identities are given in Appendix A.1. The index T in a tensor expression denotes contraction with T , for example $\bar{\nabla}_T A_\alpha = T^\beta \bar{\nabla}_\beta A_\alpha$.

The second fundamental form k_{ij} of M_t is given by $k_{ij} = -\frac{1}{2}(\mathcal{L}_T \bar{g})_{ij}$. In terms of the Fermi-propagated frame $\{e_i\}$ we have the following relations between N, T and k_{ij} .

$$\bar{\nabla}_i e_j = \nabla_i e_j - k_{ij} T, \quad \bar{\nabla}_i T = -k_{ij} e_j, \quad (2.2a)$$

$$\bar{\nabla}_T e_i = (N^{-1} \nabla_i N) T, \quad \bar{\nabla}_T T = (N^{-1} \nabla_i N) e_i. \quad (2.2b)$$

In computations we frequently make use of equations (2.2) to do an $n+1$ split, for example $\bar{\nabla}_i A_j = \nabla_i A_j + k_{ij} A_T$.

2.1. The vacuum Einstein equations. The vacuum Einstein equations

$$\bar{R}_{\alpha\beta} = 0, \quad (2.3)$$

consist after a $n+1$ split of the constraint equations

$$R - |k|^2 + (\text{tr}k)^2 = 0, \quad (2.4a)$$

$$\nabla_i \text{tr}k - \nabla^j k_{ij} = 0, \quad (2.4b)$$

and the evolution equations

$$\mathcal{L}_{\partial_t} g_{ij} = -2Nk_{ij} + \mathcal{L}_X g_{ij}, \quad (2.5a)$$

$$\mathcal{L}_{\partial_t} k_{ij} = -\nabla_i \nabla_j N + N(R_{ij} + \text{tr}k k_{ij} - 2k_{im} k_j^m) + \mathcal{L}_X k_{ij}. \quad (2.5b)$$

We will call a solution (g^0, k^0) to the Einstein vacuum constraint equations on M , a **vacuum data set**. A curve $t \mapsto (g, k, N, X)$ solving the Einstein vacuum evolution and constraint equations corresponds to a vacuum space-time metric \bar{g} via (2.1). A vacuum space-time (\bar{M}, \bar{g}) with an isometric imbedding of a vacuum data set (g, k) on M is said to be a **vacuum extension** of (g, k) .

Let \hat{g} be a fixed C^∞ Riemann metric on M with Levi-Civita covariant derivative $\hat{\nabla}$ and Christoffel symbol $\hat{\Gamma}_{ij}^k$. Define the vector field V^k by

$$V^k = g^{ij} e^k (\nabla_i e_j - \hat{\nabla}_i e_j). \quad (2.6)$$

In terms of a coordinate frame, $V^k = g^{ij} (\Gamma_{ij}^k - \hat{\Gamma}_{ij}^k)$. The identity map $\text{Id} : (M, g) \rightarrow (M, \hat{g})$ is harmonic exactly when $V^k = 0$, see [1] for discussion.

A vacuum data set (g, k) is in **CMCSH gauge** with respect to \hat{g} if

$$\text{tr}_g k = t \quad (\text{Constant Mean Curvature}), \quad (2.7a)$$

$$V^k = 0 \quad (\text{Spatial Harmonic coordinates}), \quad (2.7b)$$

Given a space-time (\bar{M}, \bar{g}) , a foliation $\{M_t, t \in (T_-, T_+)\}$ in (\bar{M}, \bar{g}) is called a CMC foliation if $\nabla \text{tr}k = 0$ for all $t \in (T_-, T_+)$. In this case, we may after a change of time parameter assume $t = \text{tr}k$. If the induced data (g, k) on M_t is in CMCSH gauge for all $t \in (T_-, T_+)$, then $\{M_t\}$ is called a CMCSH foliation. The CMCSH gauge conditions imply the following elliptic equations for the lapse and shift

$$-\Delta N + |k|^2 N = 1, \quad (2.8a)$$

$$\begin{aligned} \Delta X^i + R^i_f X^f - \mathcal{L}_X V^i &= (-2Nk^{mn} + 2\nabla^m X^n) e^i (\nabla_m e_n - \hat{\nabla}_m e_n) \\ &\quad + 2\nabla^m N k_m^i - \nabla^i N k_m^m, \end{aligned} \quad (2.8b)$$

where $\Delta X^i = g^{mn} \nabla_m \nabla_n X^i$. The ellipticity constant $\Lambda[g]$ of g is defined as the least $\Lambda \geq 1$ so that

$$\Lambda^{-1} g(Y, Y) \leq \hat{g}(Y, Y) \leq \Lambda g(Y, Y), \quad \forall Y \in TM. \quad (2.9)$$

Let \bar{g} defined in terms of g, N, X by (2.1). Define $\Lambda[\bar{g}]$ by

$$\Lambda[\bar{g}] = \Lambda[g] + \|N\|_{L^\infty} + \|N^{-1}\|_{L^\infty} + \|X\|_{L^\infty}. \quad (2.10)$$

Then \bar{g} is a nondegenerate Lorentz metric, as long as $\Lambda[\bar{g}]$ is bounded.

We refer to [1] for the background and proof of the following theorem and for the analysis concepts used in the present paper.

Theorem 2.1 ([1]). *Assume that M is of hyperbolic type with hyperbolic metric \hat{g} of unit negative sectional curvature. Let $(g^0, k^0) \in H^s \times H^{s-1}$, $s > n/2 + 1$, s integer, be a vacuum data set on M in CMCSH gauge with respect to \hat{g} . Let $t_0 = \text{tr}k^0$. The following holds.*

- (1) **Existence:** *There are $T_- < t_0 < T_+ \leq 0$ so that there is a vacuum extension (\bar{M}, \bar{g}) of (g^0, k^0) , $\bar{M} = (T_-, T_+) \times M$, $\bar{g} \in H^s(\bar{M})$, and such that the foliation $\{M_t = \{t\} \times M, t \in (T_-, T_+)\}$, is CMCSH.*
- (2) **Continuation:** *Suppose that (T_-, T_+) is maximal among all intervals satisfying point 1. Then either $(T_-, T_+) = (-\infty, 0)$ or*

$$\limsup (\Lambda[\bar{g}] + \|D\bar{g}\|_{L^\infty} + \|k\|_{L^\infty}) = \infty$$

as $t \nearrow T_+$ or as $t \searrow T_-$.

- (3) **Cauchy stability:** *Let \bar{g} be the space-time metric constructed from the solution (g, k, N, X) to the Einstein vacuum equations in CMCSH gauge. The map $(g^0, k^0) \rightarrow \bar{g}$ is continuous $H^s \times H^{s-1} \rightarrow H^s((t_-, t_+) \times M)$, for all t_-, t_+ , satisfying $T_- < t_- < t_+ < T_+$.*

2.2. Hyperbolic cone space-times. Let (M, γ) be a compact manifold of hyperbolic type, of dimension $n \geq 2$, with hyperbolic metric γ of sectional curvature -1 . The **hyperbolic cone space-time** $(\bar{M}, \bar{\gamma}_0)$ with spatial section M is the Lorentzian cone over (M, γ) , i.e.

$$\bar{M} = (0, \infty) \times M, \quad \bar{\gamma} = -d\rho^2 + \rho^2\gamma.$$

Let $(\bar{M}, \bar{\gamma})$ be a hyperbolic cone spacetime of dimension $n + 1$. The family of hyperboloids M_ρ given by $\rho = \text{constant}$ has normal

$$T = \partial_\rho.$$

Here T is future directed w.r.t. the time function ρ . Construct an adapted ON frame T, e_i on \bar{M} . A calculation gives

$$k_{ij} = -\frac{1}{\rho}g_{ij},$$

and the mean curvature is given by $\text{tr}k = -n/\rho$. The mean curvature time is defined by setting $t = \text{tr}k$ and the t -foliation has lapse

$$N = -\langle \partial_t, T \rangle = \frac{n}{t^2}.$$

In terms of the mean curvature time we have

$$g(t) = \frac{n^2}{t^2}\gamma, \quad k(t) = \frac{n}{t}\gamma. \quad (2.11)$$

In the rest of this section we will consider CMCSH foliations, with the reference metric \hat{g} chosen as $\hat{g} = \gamma$.

2.3. The constraint set and the slice. Let M be a compact manifold of hyperbolic type, of dimension $n \geq 2$ with hyperbolic metric γ of sectional curvature -1 .

For $s > n/2$, let \mathcal{M}^s be the manifold of Riemann metrics of Sobolev class H^s on M . Then \mathcal{M}^s is a smooth Hilbert manifold and \mathcal{D}^{s+1} acts on \mathcal{M}^s as a Frechet Lie group.

Lemma 2.2. *Let $s > n/2 + 1$ and fix $\tau \in \mathbb{R}$, $\tau \neq 0$. There is an open neighborhood $\mathcal{U}_\tau^s \subset \mathcal{M}^s$ of $\frac{n^2}{\tau^2}\gamma$, so that for all $g \in \mathcal{U}_\tau^s$, there is a unique $\phi \in \mathcal{D}^{s+1}(M)$, so that $\phi : (M, g) \rightarrow (M, \gamma)$ is harmonic.*

Proof. M is compact and γ has negative sectional curvature. Then there is a unique harmonic map $\phi \in H^{s+1}(M; M)$ from (M, g) to (M, γ) [6]. For g close to γ , the implicit function theorem shows ϕ is close to the identity map Id and hence $\phi \in \mathcal{D}^{s+1}(M)$. \square

Let \mathcal{U}_τ^s be as in Lemma 2.2. Let $\mathcal{S}_\tau^s \subset \mathcal{M}^s$ be defined by

$$\mathcal{S}_\tau^s = \{g \in \mathcal{U}_\tau^s : \text{Id} : (M, g) \rightarrow (M, \gamma) \text{ is harmonic}\}. \quad (2.12)$$

For $g \in \mathcal{U}_\tau^s$, if ϕ is the harmonic map provided by Lemma 2.2, $\phi_*g \in \mathcal{S}_\tau^s$. By uniqueness for harmonic maps with target (M, γ) , it follows that \mathcal{S}_τ^s is a local slice for the action of \mathcal{D}^{s+1} on \mathcal{M} . For $s > n/2$, let

$$\begin{aligned} \mathcal{C}_\tau^s = \{(g, k) \in H^s \times H^{s-1}, \quad \text{tr}_g k = \tau, \\ (g, k) \text{ solves the constraint equations (2.4)}\}, \end{aligned} \quad (2.13)$$

be the set of solutions to the vacuum Einstein constraint equations, with $\text{tr} k = \tau$. As M is a manifold of hyperbolic type, \mathcal{C}_τ^s is a smooth Hilbert manifold, cf. [7]. The action of \mathcal{D}^{s+1} on \mathcal{C}_τ^s is the lift of the action on \mathcal{M} , and therefore the local slice $\mathcal{S}_\tau^s \subset \mathcal{M}^s$ lifts to a local slice Σ_τ^s , at $(\frac{n^2}{\tau^2}\gamma, k) \in \mathcal{C}_\tau^s$,

$$\Sigma_\tau^s = \{(g, k) : (g, k) \in \mathcal{C}_\tau^s \text{ and } g \in \mathcal{S}_\tau^s\}.$$

The slice Σ_τ^s is a smooth Hilbert submanifold of \mathcal{C}_τ^s .

A symmetric 2-tensor h on (M, g) , which satisfies $\text{tr} h = 0$, $\text{div} h = 0$, is called a TT-tensor (w.r.t. g). In the rest of this section, let \mathbf{D} denote the Frechet derivative in the direction $(h, p) \in T_{(\gamma, -\gamma)}\mathcal{C}_{-n}$. It is important to keep in mind that expressions involving \mathbf{D} are evaluated at $(\gamma, -\gamma)$.

Lemma 2.3.

$$T_{(\frac{n^2}{\tau^2}\gamma, \frac{n}{\tau}\gamma)}\Sigma_\tau = \{(h, p), \quad h, p \text{ TT-tensors w.r.t. } \gamma\}. \quad (2.14)$$

Proof. We give the proof assuming $\tau = -n$, the general case follows by scaling. First note

$$0 = \mathbf{D}\text{tr} k = \text{tr}_\gamma h + \text{tr}_\gamma p. \quad (2.15)$$

Since $k|_{(\gamma, -\gamma)} = -\gamma$, (2.15) implies $\mathbf{D}|k|^2 = 0$. Let $H = R + (\text{tr} k)^2 - |k|^2$, so that the Hamiltonian constraint (2.4a) is $0 = H$. By the above,

$$0 = \mathbf{D}H = \mathbf{D}R.$$

Any symmetric 2-tensor can be decomposed as

$$h = \psi\gamma + h_{\text{TT}} + \mathcal{L}_Y\gamma,$$

where ψ is a function, h_{TT} is a TT-tensor and Y is a vector-field. As γ is hyperbolic, $R[\gamma]$ is constant, and due to covariance of R , $\mathbf{D}R.\mathcal{L}_Y\gamma = YR[\gamma] = 0$. The Frechet derivative of the scalar curvature is the operator

$$\mathbf{D}R.u = -\nabla^k \nabla_k u_i^i + \nabla^i \nabla^j u_{ij} - R_{ij} u^{ij},$$

which by the above gives

$$\begin{aligned} \mathbf{D}R.h &= \mathbf{D}R.(\psi\gamma) \\ &= -(n-1)\Delta_\gamma\psi + n(n-1)\psi. \end{aligned}$$

In view of the fact that Δ is negative semidefinite, $0 = \mathbf{D}R$ implies $\psi = 0$. Thus,

$$h = h_{\text{TT}} + \mathcal{L}_Y\gamma. \quad (2.16)$$

Let V be given by (2.6). Then with $\hat{g} = \gamma$,

$$\mathbf{D}V^i = \nabla^j h_j^i - \frac{1}{2}\nabla^i \text{tr}h,$$

where $\nabla, \Gamma, \text{tr}$ are defined w.r.t. γ . By definition, $V = 0$ on \mathcal{S}_τ , and therefore $(h, p) \in T_{(\gamma, -\gamma)}\Sigma_\tau$ implies using (2.16)

$$\mathbf{D}V = \mathbf{D}V.L_Y\gamma,$$

which by the uniqueness of harmonic maps with target γ implies that $Y = 0$ and hence $h = h_{\text{TT}}$. In particular $\text{tr}h = 0$ and therefore by (2.15), $\text{tr}p = 0$.

Let $C_i = \nabla_i \text{tr}k - \nabla^j k_{ji}$ so that the momentum constraint (2.4b) is $0 = C_i$. By assumption, $\nabla_i \text{tr}k = 0$, which using $h = h_{\text{TT}}$ and the momentum constraint (2.4b) gives

$$0 = \nabla^j p_{ji},$$

where ∇ is the covariant derivative w.r.t. γ . By the above, $\text{tr}p = 0$ so p is a TT-tensor w.r.t. γ . \square

2.4. Flat space-times. Let $(\bar{M}, \bar{\gamma})$ be a hyperbolic cone space-time of dimension $n+1$, $n \geq 2$, with spatial section (M, γ) . Consider a vacuum metric \bar{g} on \bar{M} . Then, $C_{\alpha\beta\gamma\delta} = \bar{R}_{\alpha\beta\gamma\delta}$ is the Weyl tensor and by the structure equations,

$$C_{iTjT} = R_{ij} - k_{im}k_j^m + k_{ij}\text{tr}k \quad (2.17)$$

$$C_{mTij} = d^\nabla k_{mij} \quad (2.18)$$

where the covariant exterior derivative $d^\nabla u$ on symmetric 2-tensors is

$$(d^\nabla u)_{ijk} = \nabla_k u_{ij} - \nabla_j u_{ik}$$

Let $E_{ij} = C_{iTjT}$ and $F_{mij} = C_{mTij}$, considered as tensors on M , and define the second order elliptic operator A on symmetric 2-tensors by

$$Au = \nabla^* \nabla u - nu, \quad (2.19)$$

so that $(Au)_{ij} = -\nabla^k \nabla_k u_{ij} - nu_{ij}$. By [11, Lemma 4]

$$\ker A = \ker \operatorname{tr} \cap \ker d^\nabla. \quad (2.20)$$

An element of $\ker d^\nabla$ is called a **Codazzi tensor**, i.e. the kernel of A consists of the trace-free Codazzi tensors. Clearly, a trace-free Codazzi tensor is also a TT-tensor. Let \mathbf{D} denote the Frechet derivative in the direction

$$(h, p) \in T_{(\gamma, -\gamma)} \Sigma_\tau,$$

as in section 2.3.

Lemma 2.4.

$$2\mathbf{D}E(\gamma, -\gamma)(h, p) = Ah - (n-2)h - 2(n-2)p, \quad (2.21a)$$

$$\mathbf{D}F(\gamma, -\gamma)(h, p) = d^\nabla(p + h). \quad (2.21b)$$

Proof. By Lemma 2.3, (h, p) are TT-tensors w.r.t. γ . If h is a TT-tensor, w.r.t. γ , then

$$\mathbf{D}R_{ij}.h = \frac{1}{2}\nabla^* \nabla h_{ij} - nh_{ij} = -\frac{1}{2}\nabla^k \nabla_k h_{ij} - nh_{ij}, \quad (2.22)$$

which gives (2.21a) after simplification. The Frechet derivative of Γ_{jk}^i is given by

$$D\Gamma_{jk}^i.h = \frac{1}{2}g^{im}(\nabla_j h_{km} + \nabla_k h_{jm} - \nabla_m h_{jk}). \quad (2.23)$$

A computation using (2.23) and (2.18) yields

$$\mathbf{D}C_{mTij} = \nabla_j h_{im} - \nabla_i h_{jm} + \nabla_j p_{im} - \nabla_i p_{jm},$$

which gives (2.21b). \square

Consider a curve \bar{g}_λ of vacuum metrics on \bar{M} , $\bar{g}_0 = \bar{\gamma}$, such that $\{M_t\}$ is CMCSH foliation with respect to \bar{g}_λ , and let g_λ, k_λ be the induced data on M_{-n} . Then

$$(h, p) = \left. \frac{\partial}{\partial \lambda} (g_\lambda, k_\lambda) \right|_{\lambda=0},$$

satisfy $(h, p) \in T_{(\gamma, -\gamma)} \Sigma_{-n}$. As above, let \mathbf{D} denote the Frechet derivative in the direction (h, p) . If we further assume that \bar{g}_λ is a family of flat metrics, then $\mathbf{D}E = 0$ and $\mathbf{D}F = 0$.

Decompose h, p using the L^2 -orthogonal direct sum decomposition $\ker A \oplus \ker^T A$, and write $h = h^0 + h^1$, $p = p^0 + p^1$, with $h^0, p^0 \in \ker A$, $h^1, p^1 \in \ker^T A$. Then $\mathbf{D}E = 0$ is equivalent to the system of equations

$$(n-2)[h^0 + 2p^0] = 0, \quad (2.24)$$

$$Ah^1 - (n-2)h^1 - 2(n-2)p^1 = 0. \quad (2.25)$$

By Lemma 2.3, h, p are TT-tensors on (M, γ) . Therefore, by (2.21b), $h+p \in \ker \operatorname{tr} \cap \ker d^\nabla$, and hence, by (2.20), $h^1 + p^1 = 0$. This means that equation (2.25) is equivalent to

$$0 = Ah^1 + (n-2)h^1.$$

The restriction of A to $\ker^T A$ is positive definite, so it follows that $h^1 = 0$. However, we know that $h^1 + p^1 = 0$, and hence $h^1 = p^1 = 0$. Thus we have shown that $(h, p) = (h^0, p^0)$. It remains to make use of (2.24). In case $n = 2$, this is trivial, while if $n \geq 3$, $h^0 + 2p^0 = 0$ follows.

By construction, $\ker \mathbf{DE} \cap \ker \mathbf{DF}$ is precisely the formal tangent space $T_{\bar{\gamma}}\mathbb{F}(\bar{M})$ at $\bar{\gamma}$, of the space of flat Lorentz metrics $\mathbb{F}(\bar{M})$ on \bar{M} . Recalling that in dimension 2, TT-tensors are precisely trace-free Codazzi tensors [2], we have proved

Lemma 2.5. *If $n = 2$, $T_{\bar{\gamma}}\mathbb{F}(\bar{M})$ is isomorphic to the direct sum of the space of TT-tensors on M with itself, while for $n \geq 3$, $T_{\bar{\gamma}}\mathbb{F}(\bar{M})$ is isomorphic to the space of trace-free Codazzi tensors on M . \square*

In case $n = 2$, M is a Riemann surface of genus ≥ 2 , and in this case $T_{\bar{\gamma}}\mathbb{F}(\bar{M})$ has dimension $12\text{genus}(M) - 12$, while for $n \geq 3$, $T_{\bar{\gamma}}\mathbb{F}(\bar{M})$ is trivial in case (M, γ) has no non-vanishing trace-free Codazzi tensors, a topological condition. This motivates the following definition.

Definition 2.6. *A hyperbolic manifold (M, γ) of dimension 3, is **rigid** if it admits no non-zero Codazzi tensors with vanishing trace. A hyperbolic cone space-time $(\bar{M}, \bar{\gamma})$ is called **rigid** if (M, γ) is rigid.*

A computation [11] shows that (M, γ) is rigid in the sense of Definition 2.6, if and only if the formal tangent space at γ , of the space of flat conformal structures on M is trivial. Kapovich [10, Theorem 2] proved the existence of compact hyperbolic 3-manifolds which are rigid w.r.t. infinitesimal deformations in the space of flat conformal structures. We formulate this as

Proposition 2.7. *The class of rigid hyperbolic 3-manifolds (M, γ) (and rigid standard space-times $(\bar{M}, \bar{\gamma})$), in the sense of Definition 2.6, is non-empty. \square*

3. WEYL FIELDS

In this section, and in the rest of the paper, let $n = 3$. A tracefree 4-tensor W with the symmetries of the Riemann tensor is called a Weyl field. We define the left and right Hodge duals of W by

$${}^*W_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}W^{\mu\nu}_{\gamma\delta}, \quad (3.1)$$

$$W^*_{\alpha\beta\gamma\delta} = W_{\alpha\beta}{}^{\mu\nu}\frac{1}{2}\epsilon_{\mu\nu\gamma\delta}. \quad (3.2)$$

If W is a Weyl field, then ${}^*W = W^*$ and $W = -*({}^*W)$. Define the tensors J and J^* by

$$\bar{\nabla}^\alpha W_{\alpha\beta\gamma\delta} = J_{\beta\gamma\delta}, \quad (3.3a)$$

$$\bar{\nabla}^a {}^*W_{\alpha\beta\gamma\delta} = J^*_{\beta\gamma\delta}. \quad (3.3b)$$

Then

$$J_{\beta\gamma\delta}^* = \frac{1}{2} J_{\beta}^{\mu\nu} \epsilon_{\mu\nu\gamma\delta},$$

and

$$\bar{\nabla}_{[\mu} W_{\gamma\delta]\alpha\beta} = \frac{1}{3} \epsilon_{\nu\mu\gamma\delta} J_{\alpha\beta}^{*\nu}, \quad (3.4a)$$

$$\bar{\nabla}_{[e} {}^* W_{\gamma\delta]\alpha\beta} = -\frac{1}{3} \epsilon_{\nu\mu\gamma\delta} J_{\alpha\beta}^{\nu}. \quad (3.4b)$$

The electric and magnetic parts $E(W)$, $B(W)$ of the Weyl field W , with respect to the foliation M_t are defined by

$$E(W)_{\alpha\beta} = W_{\alpha\mu\beta\nu} T^{\mu} T^{\nu}, \quad B(W)_{\alpha\beta} = {}^* W_{\alpha\mu\beta\nu} T^{\mu} T^{\nu}. \quad (3.5)$$

The tensors E and B are t -tangent, i.e. $E_{\alpha\beta} T^{\beta} = B_{\alpha\beta} T^{\beta} = 0$ and tracefree, $\bar{g}^{\alpha\beta} E_{\alpha\beta} = \bar{g}^{\alpha\beta} B_{\alpha\beta} = 0$. It follows that $g^{ij} E_{ij} = g^{ij} B_{ij} = 0$.

In case (\bar{M}, \bar{g}) is vacuum, i.e. $\bar{R}_{\alpha\beta} = 0$, the Weyl tensor $C_{\alpha\beta\gamma\delta}$ of (\bar{M}, \bar{g}) satisfies $C_{\alpha\beta\gamma\delta} = \bar{R}_{\alpha\beta\gamma\delta}$ the Gauss and Codazzi equations can be written in terms of E and B to give

$$\nabla_i k_{jm} - \nabla_j k_{im} = \epsilon_{ij}^{l} B(C)_{lm}, \quad (3.6a)$$

$$R_{ij} - k_{im} k_j^m + k_{ij} \text{tr} k = E(C)_{ij}. \quad (3.6b)$$

Note that from the definition (A.18) of d^{∇} , (3.6a) is equivalent to $d^{\nabla} k_{mij} = -\epsilon_{ij}^{l} B_{lm}$. Using the definition (A.13) of curl and (A.19) we get the alternate form of (3.6a), valid if (g, k) satisfies the vacuum constraint equations (2.4),

$$-(\text{curl} k)_{ij} = B(C)_{ij}. \quad (3.7)$$

The following identities relate $W, {}^* W, E = E(W), B = B(W)$, cf. [5, eq. (7.2.1), p. 169]

$$\begin{aligned} W_{ijkT} &= -\epsilon_{ij}^{m} B_{mk}, & {}^* W_{ijkT} &= \epsilon_{ij}^{m} E_{mk}, \\ W_{ijk\ell} &= -\epsilon_{ijm} \epsilon_{k\ell n} E^{mn}, & {}^* W_{ijk\ell} &= -\epsilon_{ijm} \epsilon_{k\ell n} B^{mn}. \end{aligned} \quad (3.8)$$

The tensors $\bar{\nabla}_T E$, $\bar{\nabla}_T B$ have the property

$$g^{ij} \bar{\nabla}_T E_{ij} = 0, \quad g^{ij} \bar{\nabla}_T B_{ij} = 0,$$

i.e. the pull-back of $\bar{\nabla}_T E$, $\bar{\nabla}_T B$ to M is trace-free, but $\bar{\nabla}_T E$, $\bar{\nabla}_T B$ are not t -tangent in general. The following result allows us to express derivatives of the Weyl field W in terms of $E(W), B(W), J(W), J^*(W)$. See Appendix A.2 for the definition of div and curl.

Proposition 3.1. *Let E, B be the electric and magnetic parts of a Weyl-field W and let J, J^* be defined from W by (3.4). Then*

$$\text{div} E_i = +(k \wedge B)_i + J_{TiT}, \quad (3.9a)$$

$$\text{div} B_i = -(k \wedge E)_i + J_{TiT}^*. \quad (3.9b)$$

$$\bar{\nabla}_T E_{ij} - \text{curl} B_{ij} = -N^{-1}(\nabla N \wedge B)_{ij} - \frac{3}{2}(E \times k)_{ij} + \frac{1}{2}(\text{tr} k)E_{ij} - J_{iTj}, \quad (3.10a)$$

$$\bar{\nabla}_T B_{ij} + \text{curl} E_{ij} = +N^{-1}(\nabla N \wedge E)_{ij} - \frac{3}{2}(B \times k)_{ij} + \frac{1}{2}(\text{tr} k)B_{ij} - J_{iTj}^*. \quad (3.10b)$$

Written in terms of \mathcal{L}_{∂_t} , (3.10) becomes

$$\begin{aligned} N^{-1}\mathcal{L}_{\partial_t} E_{ij} = & +\text{curl} B_{ij} - N^{-1}(\nabla N \wedge B)_{ij} \\ & - \frac{5}{2}(E \times k)_{ij} - \frac{2}{3}(E \cdot k)g_{ij} - \frac{1}{2}\text{tr} k E_{ij} + N^{-1}\mathcal{L}_X E_{ij} - J_{iTj}, \end{aligned} \quad (3.11a)$$

$$\begin{aligned} N^{-1}\mathcal{L}_{\partial_t} B_{ij} = & -\text{curl} E_{ij} + N^{-1}(\nabla N \wedge E)_{ij} \\ & - \frac{5}{2}(B \times k)_{ij} - \frac{2}{3}(B \cdot k)g_{ij} - \frac{1}{2}(\text{tr} k)B_{ij} + N^{-1}\mathcal{L}_X B_{ij} - J_{iTj}^*. \end{aligned} \quad (3.11b)$$

Proof. We write $\bar{\nabla}^\alpha E_{\alpha i}$ in two ways. First,

$$\text{div} E_i = \bar{\nabla}^\alpha E_{\alpha i} - N^{-1}\nabla^j N E_{ji}.$$

Secondly, by (3.8) and (A.10),

$$\begin{aligned} \bar{\nabla}^\alpha E_{\alpha i} = & \bar{\nabla}^\alpha W_{\alpha T i T} + \bar{g}^{\alpha\beta} W_{\alpha\gamma i\delta} \bar{\nabla}_\beta T^\gamma T^\delta + \bar{g}^{\alpha\beta} W_{\alpha\gamma i\delta} T^\gamma \bar{\nabla}_\beta T^\delta \\ = & J_{TiT} + (k \wedge B)_i + N^{-1}\nabla^j N E_{ji}. \end{aligned}$$

This gives (3.9a) and the argument for (3.9b) is similar. To prove (3.10), first note the identities

$$\bar{\nabla}_k W_{iTjT} = \nabla_k E_{ij} - (\epsilon_{il}{}^m B_{mj} + \epsilon_{jl}{}^m B_{mi})k^l{}_k, \quad (3.12a)$$

$$\bar{\nabla}_k {}^*W_{iTjT} = \nabla_k B_{ij} + (\epsilon_{il}{}^m E_{mj} + \epsilon_{jl}{}^m E_{mi})k^l{}_k. \quad (3.12b)$$

From this we get, after expanding the covariant derivative, and rewriting using (3.8)

$$\epsilon_i{}^{mn} \bar{\nabla}_n W_{mTjT} + \epsilon_j{}^{mn} \bar{\nabla}_n W_{mTiT} = 2(\text{curl} E)_{ij} + 3(B \times k)_{ij} - (\text{tr} k)B_{ij}, \quad (3.13a)$$

$$\epsilon_i{}^{mn} \bar{\nabla}_n {}^*W_{mTjT} + \epsilon_j{}^{mn} \bar{\nabla}_n {}^*W_{mTiT} = 2(\text{curl} B)_{ij} - 3(E \times k)_{ij} + (\text{tr} k)E_{ij}. \quad (3.13b)$$

The Bianchi equations (3.4) imply

$$\epsilon_i{}^{mn} \bar{\nabla}_T W_{mnjT} = 2\epsilon_i{}^{mn} \bar{\nabla}_n W_{mTjT} - 2J_{ijT}^*, \quad (3.14a)$$

$$\epsilon_i{}^{mn} \bar{\nabla}_T {}^*W_{mnjT} = 2\epsilon_i{}^{mn} \bar{\nabla}_n {}^*W_{mTjT} + 2J_{ijT}. \quad (3.14b)$$

Using (3.8) and (2.2) we get

$$\epsilon_i{}^{mn} \bar{\nabla}_T W_{mnjT} = -2\bar{\nabla}_T B_{ij} + 2N^{-1}(\nabla N \wedge E)_{ij}, \quad (3.15a)$$

$$\epsilon_i{}^{mn} \bar{\nabla}_T {}^*W_{mnjT} = 2\bar{\nabla}_T E_{ij} + 2N^{-1}(\nabla N \wedge B)_{ij}. \quad (3.15b)$$

Using (3.15), multiplying by $\frac{1}{2}$, taking the symmetric parts of (3.14) and using (3.13) now gives the identities (3.10). It is straightforward to derive (3.11) from (3.10) using (A.17). \square

Given a Weyl field W , the covariant derivative $\bar{\nabla}_T W$ is again a Weyl field. Proposition 3.1 gives the following expressions for $E(\bar{\nabla}_T W), B(\bar{\nabla}_T W)$.

Corollary 3.2.

$$E(\bar{\nabla}_T W)_{ij} = +\text{curl} B_{ij} - \frac{3}{2}(E \times k)_{ij} + \frac{1}{2}(\text{tr} k)E_{ij} - J_{iTj}, \quad (3.16a)$$

$$B(\bar{\nabla}_T W)_{ij} = -\text{curl} E_{ij} - \frac{3}{2}(B \times k)_{ij} + \frac{1}{2}(\text{tr} k)B_{ij} - J_{iTj}^*. \quad (3.16b)$$

where in the right hand side, E, B, J, J^* are defined w.r.t. W .

Proof. From the definition and using (2.2) we have, taking into account the fact that E is t -tangent,

$$\begin{aligned} E(\bar{\nabla}_T W)_{ij} &= T^\gamma T^\delta T^\nu \bar{\nabla}_\nu W_{i\gamma j\delta} \\ &= T^\nu \bar{\nabla}_\nu E(W)_{ij} - N^{-1} \nabla^m N W_{imjT} - N^{-1} \nabla^n N W_{iTjn} \end{aligned}$$

using (3.8) and (A.11)

$$= T^\nu \bar{\nabla}_\nu E(W)_{ij} + N^{-1}(\nabla N \wedge B(W))_{ij},$$

which gives (3.16a) using (3.10a). The proof of (3.16b) is similar. \square

4. THE BEL-ROBINSON ENERGY

Given a Weyl field W we can associate to it a fully symmetric and traceless tensor

$$Q(W)_{\alpha\beta\gamma\delta} = W_{\alpha\mu\gamma\nu} W_{\beta}^{\mu\nu} + {}^*W_{\alpha\mu\gamma\nu} {}^*W_{\beta}^{\mu\nu}. \quad (4.1)$$

$Q(W)$ is positive definite in the sense that $Q(X, Y, X, Y) \geq 0$ whenever X, Y are timelike vectors, with equality only if W vanishes, cf. [4, Prop. 4.2] Let $E = E(W), B = B(W)$. The following identities relate $Q(W)$ to E and B .

$$Q(W)_{TTTT} = |E|^2 + |B|^2, \quad (4.2a)$$

$$Q(W)_{iTTT} = 2(E \wedge B)_i, \quad (4.2b)$$

$$Q(W)_{ijTT} = -(E \times E)_{ij} - (B \times B)_{ij} + \frac{1}{3}(|E|^2 + |B|^2)g_{ij}, \quad (4.2c)$$

where $|E|^2 = E^{ij}E_{ij} = |E|_g^2$, and similarly for $|B|^2$. From equations (3.8) and (4.2a) it follows that $Q(W)_{TTTT} = 0$ if and only if $W = 0$. The divergence of the Bel–Robinson tensor takes the form [5, Prop. 7.1.1]

$$\begin{aligned} \bar{\nabla}^\alpha Q(W)_{\alpha\beta\gamma\delta} &= W_{\beta}^{\mu\nu} J(W)_{\mu\gamma\nu} + W_{\beta}^{\mu\gamma} J(W)_{\mu\delta\nu} \\ &\quad + {}^*W_{\beta}^{\mu\nu} J^*(W)_{\mu\gamma\nu} + {}^*W_{\beta}^{\mu\gamma} J^*(W)_{\mu\delta\nu}, \end{aligned} \quad (4.3)$$

and the definition of $E(W)$ and $B(W)$ gives

$$\bar{\nabla}^\alpha Q(W)_{\alpha TTT} = 2E^{ij}(W)J(W)_{iTj} + 2B^{ij}(W)J^*(W)_{iTj}. \quad (4.4)$$

Let W be a Weyl field and let $Q(W)$ be the corresponding Bel-Robinson tensor. Then working in a foliation M_t , we define the Bel-Robinson energy $\mathcal{Q}(t, W)$ by

$$\mathcal{Q}(t, W) = \int_{M_t} Q(W)_{TTTT} d\mu_{M_t}.$$

By the Gauss law, this has the evolution equation

$$\begin{aligned} \partial_t \mathcal{Q}(t, W) = & - \int_{M_t} N \bar{\nabla}^\alpha Q(W)_{\alpha TTT} d\mu_{M_t} \\ & - 3 \int_{M_t} N Q(W)_{\alpha\beta TT} \pi^{\alpha\beta} d\mu_{M_t}, \end{aligned} \quad (4.5)$$

where π denotes the “deformation tensor” of T ,

$$\pi_{\alpha\beta} = \bar{\nabla}_\alpha T_\beta. \quad (4.6)$$

The components of π in terms of an adapted, Fermi-propagated frame are as follows:

$$\pi_{ij} = -k_{ij}, \quad \pi_{iT} = 0, \quad (4.7a)$$

$$\pi_{Ti} = N^{-1} \nabla_i N, \quad \pi_{TT} = 0. \quad (4.7b)$$

We will need control of \bar{g} in H^3 , and for this purpose we consider in addition to the Bel-Robinson energy of order zero, $\mathcal{Q}_0(t, W) = \mathcal{Q}(t, W)$, the first order Bel-Robinson energy $\mathcal{Q}_1(t, W) = \mathcal{Q}(t, \bar{\nabla}_T W)$. In the vacuum case, $J(W) = J^*(W) = 0$, so we may view \mathcal{Q}_1 as a function on the set of solutions to the Einstein vacuum constraint equations, by using Corollary 3.2 to compute $E(\bar{\nabla}_T W), B(\bar{\nabla}_T W)$.

Expanding $\partial_t \mathcal{Q}(t, W)$ using (4.2), (4.4), (4.7), gives

$$\begin{aligned} \partial_t \mathcal{Q}(t, W) = & -3 \int_{M_t} N[(E \times E) \cdot k + (B \times B) \cdot k - \frac{1}{3}(|E|^2 + |B|^2) \text{tr} k \\ & - 2N^{-1} \nabla^i N(E \wedge B)_i] d\mu_{M_t} \\ & - 2 \int_{M_t} N(E^{ij} J_{iTj} + B^{ij} J_{iTj}^*) d\mu_{M_t} \end{aligned}$$

(perform a partial integration and use (A.15))

$$\begin{aligned} = & -3 \int_{M_t} N[(E \times E) \cdot k + (B \times B) \cdot k - \frac{1}{3}(|E|^2 + |B|^2) \text{tr} k \\ & - 2 \text{curl} E \cdot B + 2E \cdot \text{curl} B] d\mu_{M_t} \\ & - 2 \int_{M_t} N(E^{ij} J_{iTj} + B^{ij} J_{iTj}^*) d\mu_{M_t}. \end{aligned} \quad (4.8)$$

It is straightforward to show that this expression agrees with that obtained after a direct computation of $\partial_t \mathcal{Q}(t, W)$ using (3.11).

Let $\tau = \text{tr} k$ and specialize to a constant mean curvature foliation $\{M_\tau\}$ in the following. With the discussion in subsection 2.2 as a guide we introduce the following quantities which vanish when evaluated in the standard

foliation on a hyperbolic cone–time, namely the “trace free” part $\hat{\pi}$ of π ,

$$\hat{\pi}_{\alpha\beta} = \pi_{\alpha\beta} + \frac{\tau}{3}(\bar{g}_{\alpha\beta} + T_\alpha T_\beta) \quad (4.9)$$

and the “perturbed” part of the lapse,

$$\hat{N} = N - \frac{3}{\tau^2}. \quad (4.10)$$

In the following Lemma we record the form of $\partial_\tau \mathcal{Q}_i$, $i = 0, 1$ which will be used in the global existence proof.

Lemma 4.1. *In a vacuum space time, the Bel–Robinson energies $\mathcal{Q}_j(t, W)$, $j = 0, 1$ satisfy the following conservation laws.*

$$\begin{aligned} \partial_\tau \mathcal{Q}_0(\tau, W) &= \frac{3}{\tau} \mathcal{Q}_0(t, W) - 3 \int_{M_\tau} N Q(W)_{\alpha\beta\gamma\delta} \hat{\pi}^{\alpha\beta} T^\gamma T^\delta d\mu_{M_\tau} \\ &\quad + \tau \int_{M_\tau} \hat{N} Q(W)_{TTTT} d\mu_{M_\tau}, \end{aligned} \quad (4.11)$$

$$\begin{aligned} \partial_\tau \mathcal{Q}_1(\tau, W) &= \frac{5}{\tau} \mathcal{Q}_1(t, W) - 2 \int_{M_\tau} N \mathcal{G}_1(W) d\mu_{M_\tau} \\ &\quad - 3 \int_{M_t} N Q(\bar{\nabla}_T W)_{\alpha\beta\gamma\delta} \hat{\pi}^{\alpha\beta} T^\gamma T^\delta d\mu_{M_\tau} \\ &\quad + \frac{5\tau}{3} \int_{M_\tau} \hat{N} Q(\bar{\nabla}_T W)_{TTTT} d\mu_{M_\tau}, \end{aligned} \quad (4.12)$$

where

$$\begin{aligned} \mathcal{G}_1(W) &= E(\bar{\nabla}_T W)^{ij} (J(\bar{\nabla}_T W)_{iTj} + \frac{\tau}{3} E(\bar{\nabla}_T W)_{ij}) \\ &\quad + B(\bar{\nabla}_T W)^{ij} (J^*(\bar{\nabla}_T W)_{iTj} + \frac{\tau}{3} B(\bar{\nabla}_T W)_{ij}). \end{aligned} \quad (4.13)$$

In particular,

$$J(\bar{\nabla}_T W)_{iTj} + \frac{\tau}{3} E(\bar{\nabla}_T W)_{ij} = \hat{\pi}^{\alpha\mu} \bar{\nabla}_\mu W_{\alpha iTj} + \frac{3}{2} (E \times E)_{ij} - \frac{3}{2} (B \times B)_{ij} \quad (4.14a)$$

$$J^*(\bar{\nabla}_T W)_{iTj} + \frac{\tau}{3} B(\bar{\nabla}_T W)_{ij} = \hat{\pi}^{\alpha\mu} \bar{\nabla}_\mu^* W_{\alpha iTj} + 3(E \times B)_{ij} \quad (4.14b)$$

where

$$\begin{aligned}\hat{\pi}^{\alpha\mu}\bar{\nabla}_\mu W_{\alpha iTj} &= \hat{k}^{rs}\nabla_s(-\epsilon_{ri}{}^n B_{nj}) \\ &\quad - \hat{k}^{rs}k_{rs}E_{ij} + \hat{k}^{rs}k_{si}E_{rj} + \hat{k}^{rs}k_s{}^m\epsilon_{ri}{}^n\epsilon_{mj}{}^p E_{np} \\ &\quad - N^{-1}\nabla^r N(\nabla_r E_{ij} + k_r{}^s\epsilon_{si}{}^n B_{nj} + k_r{}^s\epsilon_{sj}{}^n B_{ni})\end{aligned}\quad (4.15a)$$

$$\begin{aligned}\hat{\pi}^{\alpha\mu}\bar{\nabla}_\mu {}^*W_{\alpha iTj} &= \hat{k}^{rs}\nabla_s(\epsilon_{ri}{}^n E_{nj}) \\ &\quad - \hat{k}^{rs}k_{rs}B_{ij} + \hat{k}^{rs}k_{si}B_{rj} + \hat{k}^{rs}k_s{}^m\epsilon_{ri}{}^n\epsilon_{mj}{}^p B_{np} \\ &\quad - N^{-1}\nabla^r N(\nabla_r B_{ij} - k_r{}^s\epsilon_{si}{}^n E_{nj} - k_r{}^s\epsilon_{sj}{}^n E_{ni})\end{aligned}\quad (4.15b)$$

Remark 4.1. In the proof of the main theorem, it is of central importance that the terms in (4.14) are quadratic in $\hat{\pi}, \bar{\nabla}W, W$. This has the consequence that the term given by (4.13) can be treated as a perturbation term in case of small data. In particular the terms $\hat{\pi}^{\alpha\mu}\bar{\nabla}_\mu W_{\alpha iTj}$ and $\hat{\pi}^{\alpha\mu}\bar{\nabla}_\mu {}^*W_{\alpha iTj}$ when expanded are seen to be of third order in $N^{-1}\nabla_i N, k_{ij}, E_{ij}, B_{ij}$ and of second order in $\hat{k}_{ij}, \nabla_i E_{jk}, \nabla_i B_{jk}$. We will not make use of the explicit expression for $\partial_t \mathcal{Q}_1$, but for completeness, it is given in equation (4.19) below.

Proof. In order to evaluate $\text{Div}Q(\bar{\nabla}_T W)(T, T, T)$, we need $J(\bar{\nabla}_T W)_{iTj}$ and $J^*(\bar{\nabla}_T W)_{iTj}$. A computation gives

$$\begin{aligned}J(\bar{\nabla}_T W)_{\beta\gamma\delta} &= \bar{\nabla}^\alpha \bar{\nabla}_T W_{\alpha\beta\gamma\delta} = \pi^{\alpha\nu}\bar{\nabla}_\nu W_{\alpha\beta\gamma\delta} + T^\nu \bar{\nabla}_\nu J(W)_{\beta\gamma\delta} \\ &\quad + T^\nu \bar{R}_\alpha{}^\mu{}_\nu W_{\mu\beta\gamma\delta} + T^\nu \bar{R}_\beta{}^\mu{}_\nu W_{\alpha\mu\gamma\delta} \\ &\quad + T^\nu \bar{R}_\gamma{}^\mu{}_\nu W_{\alpha\beta\mu\delta} + T^\nu \bar{R}_\delta{}^\mu{}_\nu W_{\alpha\beta\gamma\mu},\end{aligned}\quad (4.16)$$

Note that in vacuum, $J(W) = 0$ and $\bar{R}_{\alpha\beta\gamma\delta} = W_{\alpha\beta\gamma\delta}$. Substituting \bar{R} for W in (4.16) and using (3.8) to rewrite the terms quadratic in W gives (4.14a). A similar calculation for $J^*(\bar{\nabla}_T W)$, taking into account the fact that in this case, $\bar{R} = W$ and *W are distinct, gives (4.14b). It is now straightforward to check that (4.11) and (4.12) hold, given the definition of \mathcal{G}_1 in (4.13). \square

One can decompose equations (4.14) into symmetric and antisymmetric parts to obtain the analogues of equations (3.9) and (3.10). Setting $J(W) = J^*(W) = 0$ for the vacuum case, defining $\tilde{E}_{ij} = E(\bar{\nabla}_T W)_{ij}$ and $\tilde{B}_{ij} = B(\bar{\nabla}_T W)_{ij}$ and writing $E_{ij|k}$ and $B_{ij|k}$ for $\nabla_k E_{ij}$ and $\nabla_k B_{ij}$ respectively we get, first for the symmetric parts (the analogues of equations (3.11)),

$$\begin{aligned}N^{-1}\mathcal{L}_{\partial_t}\tilde{E}_{ij} &= k_{ij}(k^{lm}E_{lm}) + g_{ij}[E^{lm}E_{lm} + k^{ls}k_s{}^m E_{lm} + k^{lm}\tilde{E}_{lm} - B^{lm}B_{lm}] \\ &\quad + 2E_{ij}(k^{lm}k_{lm}) - 3E_{il}E_j{}^l - 3k_i{}^m\tilde{E}_{mj} \\ &\quad - 3k_j{}^m\tilde{E}_{mi} - k_i{}^m k_j{}^l E_{lm} + 3B_{il}B_j{}^l \\ &\quad + \frac{1}{2}g_{jk}\epsilon^{lkm}(\tilde{B}_{im|l} - k_{ms}B_{i|l}^s) + \frac{1}{2}g_{ik}\epsilon^{lkm}(\tilde{B}_{jm|l} - k_{ms}B_{j|l}^s) \\ &\quad + N^{-1}N^{|l}E_{ij|l} - \frac{5}{2}k_j{}^s k_s{}^m E_{im} - \frac{5}{2}k_i{}^s k_s{}^m E_{jm}\end{aligned}$$

$$\begin{aligned}
& + (\text{tr}k)[3\tilde{E}_{ij} + \frac{5}{2}k_i^m E_{jm} + \frac{5}{2}k_j^m E_{im} - 2(\text{tr}k)E_{ij} - g_{ij}k^{lm}E_{lm}] \\
& + N^{-1}N^{|l}[B_i^r \epsilon_{kjr} k_l^k + B_j^r \epsilon_{kir} k_l^k - \epsilon_{rjl} \tilde{B}_i^r - \epsilon_{ril} \tilde{B}_j^r] \\
& + N^{-1}\mathcal{L}_X \tilde{E}_{ij}
\end{aligned} \tag{4.17a}$$

$$\begin{aligned}
N^{-1}\mathcal{L}_{\partial_t} \tilde{B}_{ij} & = 2B_{ij}k^{lm}k_{lm} + g_{ij}[2B^{lm}E_{lm} + k^{ls}k_s^m B_{lm} + k^{lm}\tilde{B}_{lm}] \\
& + N^{-1}N^{|l}[B_{ij|l} - 3B_j^m E_{im} - 3B_i^m E_{jm} - 3k_i^l \tilde{B}_{lj} \\
& - 3k_j^l \tilde{B}_{li} - k_i^l k_j^s B_{ls} + k_{ij}k^{lm}B_{lm} \\
& - \frac{5}{2}(k_{jm}k^{ml}B_{li} + k_{im}k^{ml}B_{lj}) \\
& - \frac{1}{2}g_{ik}\epsilon^{lkm}(\tilde{E}_{mj|l} - k_m^s E_{sj|l}) - \frac{1}{2}g_{jk}\epsilon^{lkm}(\tilde{E}_{mi|l} - k_m^s E_{si|l}) \\
& + (\text{tr}k)[3\tilde{B}_{ij} + \frac{5}{2}k_i^l B_{lj} + \frac{5}{2}k_j^l B_{li} - 2(\text{tr}k)B_{ij} - g_{ij}k^{lm}B_{lm}] \\
& - N^{-1}N^{|l}[E_i^r \epsilon_{kjr} k_l^k + E_j^r \epsilon_{kir} k_l^k - \epsilon_{rjl} \tilde{E}_i^r - \epsilon_{ril} \tilde{E}_j^r] \\
& + N^{-1}\mathcal{L}_X \tilde{B}_{ij}
\end{aligned} \tag{4.17b}$$

and then for the antisymmetric part (the analogues for equations (3.9)),

$$\tilde{E}_j^i{}^{|j} = k_j^m \tilde{B}_{mr} \epsilon^{ijr} - k_j^m E_m^i{}^{|j} + (\text{tr}k)\epsilon^{imn}k_m^s B_{sn} - k_r^i k_l^s B_{sj} \epsilon^{rlj} \tag{4.18a}$$

$$\tilde{B}_j^i{}^{|j} = -k_j^m \tilde{E}_{mr} \epsilon^{ijr} - k_j^m B_m^i{}^{|j} - (\text{tr}k)\epsilon^{imn}k_m^s E_{sn} + k_r^i k_l^s E_{sj} \epsilon^{rlj} \tag{4.18b}$$

The formula for $\partial_t \mathcal{Q}_1$ that is analogous to that given above for $\partial_t \mathcal{Q}$ is given explicitly as follows,

$$\begin{aligned}
\frac{\partial}{\partial t} \int_M \mu_g (\tilde{E}_i^\ell \tilde{E}_\ell^i + \tilde{B}_i^\ell \tilde{B}_\ell^i) & = \partial_t \mathcal{Q}_1 \\
& = 2 \int_M \left\{ N g_{jk} \epsilon^{\ell km} \left[\hat{k}_m^s (\tilde{B}^{ij} E_{si|\ell} - \tilde{E}^{ij} B_{si|\ell}) \right] \mu_g \right. \\
& + N \mu_g (\text{tr}k) \left[\frac{5}{6} \tilde{E}^{ij} \tilde{E}_{ij} + \frac{5}{6} \tilde{B}^{ij} \tilde{B}_{ij} \right] \\
& + \mu_g N^{|l} \left(\tilde{E}^{ij} E_{ij|l} + \tilde{B}^{ij} B_{ij|l} \right) \\
& + 2N \mu_g \left(\hat{k}_{mn} \hat{k}^{mn} \right) \left(\tilde{E}^{ij} E_{ij} + \tilde{B}^{ij} B_{ij} \right) \\
& - 4N \mu_g \hat{k}_j^\ell \left[\tilde{E}_\ell^i \tilde{E}_i^j + \tilde{B}_\ell^i \tilde{B}_i^j \right] \\
& + N \mu_g \left[(\tilde{E}^{ij} \hat{k}_{ij})(E^{mn} \hat{k}_{mn}) + (\tilde{B}^{ij} \hat{k}_{ij})(B^{mn} \hat{k}_{mn}) \right] \\
& \left. - 3N \mu_g \left[\tilde{E}^{ij} E_{i\ell} E_j^\ell - \tilde{E}^{ij} B_{i\ell} B_j^\ell + 2\tilde{B}^{ij} B_j^\ell E_{i\ell} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& -3N_{|\ell}\epsilon^{rj\ell}\tilde{E}_j^i\tilde{B}_{ri}\mu_g + 2N^{|\ell}\left[\tilde{E}_j^iB_{ri}\epsilon^{kjr}(\hat{k}_{\ell k} + \frac{1}{3}g_{\ell k}(\text{tr}k))\right. \\
& \left.-\tilde{B}_j^iE_{ri}\epsilon^{kjr}(\hat{k}_{\ell k} + \frac{1}{3}g_{\ell k}(\text{tr}k))\right]\mu_g \\
& -N\mu_g\hat{k}_i^m\hat{k}_j^\ell\left[\tilde{E}^{ij}E_{lm} + \tilde{B}^{ij}B_{lm}\right] \\
& \left.-5N\mu_g\hat{k}_j^s\hat{k}_s^m(\tilde{E}^{ij}E_{im} + \tilde{B}^{ij}B_{im})\right\} \tag{4.19}
\end{aligned}$$

4.1. The scale-free Bel–Robinson energy. In the rest of section 4, let $n = 3$, and assume (M, \bar{g}) is a vacuum space-time with a CMC foliation $\{M_\tau\}$ with $\tau = \text{tr}k < 0$. Let $W_{\alpha\beta\gamma\delta} = \bar{R}_{\alpha\beta\gamma\delta}$ be the Weyl tensor of (M, \bar{g}) . Then W satisfies the homogenous Bianchi identities, i.e. $J(W) = J^*(W) = 0$.

Since we will be estimating geometric quantities in terms of $\mathcal{Q}_0, \mathcal{Q}_1$ via Sobolev inequalities which depend on scale, we need scale-free versions of these energies. It follows from the definitions that the following variables are scale-free if λ has dimensions $(length)^{-1}$. Here indices refer to a coordinate frame.

$$\tilde{g}_{ab} = \lambda^2 g_{ab}, \quad \tilde{T}^\alpha = \lambda^{-1} T^\alpha, \tag{4.20a}$$

$$\tilde{k}_{ab} = \lambda k_{ab}, \quad \widetilde{\text{tr}k} = \lambda^{-1} \text{tr}k, \tag{4.20b}$$

$$\tilde{N} = \lambda^2 N, \quad \widetilde{\mu_g} = \lambda^3 \mu_g. \tag{4.20c}$$

Note that $\text{tr}k$ has dimensions of $(length)^{-1}$ while we treat spatial coordinates as dimensionless quantities. The Weyl tensor in the $(3,1)$ form is conformally invariant, and hence the Bel–Robinson tensor is also conformally invariant, and in particular scale-free. From this can be seen that the Bel–Robinson energies $\mathcal{Q}_0, \mathcal{Q}_1$ have dimensions $(length)^{-1}$ and $(length)^{-3}$ respectively, so that the expressions

$$\tilde{\mathcal{Q}}_i = \lambda^{-1-2i} \mathcal{Q}_i, \quad i = 0, 1,$$

are scale-free, i.e. $\tilde{\mathcal{Q}}_i$ is precisely given by \mathcal{Q}_i evaluated on the scale-free variables \tilde{g}, \tilde{k} etc. In the following we will use the scale factor λ , defined by

$$\lambda = \frac{|\text{tr}k|}{3} = -\frac{\text{tr}k}{3}. \tag{4.21}$$

The scale-free energy function which will be used in the proof of global existence is the sum of $\tilde{\mathcal{Q}}_0$ and $\tilde{\mathcal{Q}}_1$,

$$\mathcal{E} = \tilde{\mathcal{Q}}_0 + \tilde{\mathcal{Q}}_1. \tag{4.22}$$

It is convenient to introduce the logarithmic time $\sigma = -\ln(-\tau)$. Then $\sigma \nearrow \infty$ as $\tau \nearrow 0$. The logarithmic time σ has the property that $\partial_\sigma = -\tau \partial_\tau$ is scale-free, so that for example $\partial_\sigma \mathcal{E}$ is scale-free.

4.2. The Hessian of the Bel–Robinson energy. Let M be a compact 3-dimensional manifold of hyperbolic type and let γ be the standard hyperbolic metric on M . Let $(\frac{9}{\tau^2}\gamma, \frac{3}{\tau}\gamma)$ be data with mean curvature τ for the hyperbolic cone space–time $(\bar{M}, \bar{\gamma})$. For $s \geq 3$, let Σ_τ^s be the local slice for the action of \mathcal{D}^{s+1} on the constraint set \mathcal{C}_τ , at the hyperbolic cone data with mean curvature τ .

The energies $\tilde{\mathcal{Q}}_0, \tilde{\mathcal{Q}}_1, \mathcal{E}$ may be thought of as functions on the constraint set \mathcal{C}_τ^s by using equations (3.6) and (3.8). The following Lemma is a straightforward consequence of the Sobolev imbedding theorems.

Lemma 4.2. *The scale–free energies $\tilde{\mathcal{Q}}_0(\tau, W), \tilde{\mathcal{Q}}_1(\tau, W), \mathcal{E}(\tau, W)$, are C^∞ functions on \mathcal{C}_τ^3 and Σ_τ^3 .*

Let $\text{Hess}\tilde{\mathcal{Q}}_0(\gamma, -\gamma)$ denote the Hessian of the function $\tilde{\mathcal{Q}}_0$, evaluated at $(\gamma, -\gamma)$. A computation shows that for $(h, p) \in T_{(\gamma, -\gamma)}T\Sigma_{-n}$,

$$\text{Hess}\tilde{\mathcal{Q}}_0(\gamma, -\gamma)((h, p), (h, p)) = \frac{1}{2}\|Ah\|_{L^2}^2 + (Ap, p)_{L^2} + \frac{1}{2}\|h + 2p\|_{L^2}^2,$$

where A is given by (2.19), and $\|\cdot\|_{L^2}, (\cdot, \cdot)_{L^2}$ denote the L^2 norm and inner product defined with respect to γ . Recall that $\ker A = \{0\}$ if and only if M is rigid. It is now straightforward to prove the following Lemma.

Lemma 4.3. *Let M be a compact hyperbolic 3–manifold. The Hessian of the scale–free Bel–Robinson energy $\tilde{\mathcal{Q}}_0$, defined by equation (4.22), considered as a function on Σ_{-3} , evaluated at the standard data $(\gamma, -\gamma)$, satisfies the inequality*

$$\text{Hess}\tilde{\mathcal{Q}}_0(\gamma, -\gamma)((h, p), (h, p)) \geq C(\|h\|_{H^2}^2 + \|p\|_{H^1}^2), \quad (4.23)$$

if and only if M is rigid. The constant C depends only on the topology of M . \square

Consider a solution \bar{h} of the linearized Einstein equations on the hyperbolic cone space–time $(\bar{M}, \bar{\gamma})$. The derivative of the Weyl tensor in the direction of h ,

$$W' = DW[\bar{\gamma}]\bar{h},$$

is a Weyl field on $(\bar{M}, \bar{\gamma})$ which satisfies the homogeneous Bianchi equations,

$$J(W') = J^*(W') = 0, \quad (4.24)$$

Let $E(W'), B(W')$ be the electric and magnetic parts of W' at $M_{-3} \subset \bar{M}$. Then

$$DE(\bar{\nabla}_T W)|_{\bar{\gamma}}\bar{h} = E(\bar{\nabla}_T W'), \quad DB(\bar{\nabla}_T W)|_{\bar{\gamma}}\bar{h} = B(\bar{\nabla}_T W').$$

Recall that the second fundamental form of M_{-3} is $k = -\gamma$. This implies using (3.9), (4.24),

$$\text{div}E(\bar{\nabla}_T W') = \text{div}B(\bar{\nabla}_T W') = 0, \quad (4.25)$$

which shows that $E(\bar{\nabla}_T W')$ and $B(\bar{\nabla}_T W')$ are TT-tensors. It follows from Corollary 3.2, using $k = -\gamma$ and (4.24),

$$\begin{aligned} E(\bar{\nabla}_T W') &= +\operatorname{curl} B(W') + \frac{1}{2} \operatorname{tr} k E(W'), \\ B(\bar{\nabla}_T W') &= -\operatorname{curl} E(W') + \frac{1}{2} \operatorname{tr} k B(W'). \end{aligned}$$

The scale-free Bel–Robinson energy \mathcal{E} is a smooth function on Σ_τ^3 . Using the above it is straightforward, using (A.23) to prove that the Hessian of \mathcal{E} is positive definite on $H^3 \times H^2$ in case M is rigid. We state this as

Theorem 4.4. *The hessian $\operatorname{Hess} \mathcal{E}$ on Σ_τ , evaluated at the standard data $(\frac{\tau^2}{9}\gamma, \frac{\tau}{3}\gamma)$, satisfies the inequality*

$$\operatorname{Hess} \mathcal{E}(\frac{9}{\tau^2}\gamma, \frac{3}{\tau}\gamma)((h, p), (h, p)) \geq C(\|h\|_{H^3}^2 + \|p\|_{H^2}^2), \quad (4.26)$$

if and only if M is rigid. The constant C depend only on the topology of M .

Results analogous to Theorem 4.4 can easily be proved for even higher order Bel–Robinson type energies. This will not be needed in this paper.

5. ESTIMATES

In this section we will introduce a “smallness condition” on the vacuum data (g, k) , under which we are able to control all relevant geometric quantities in terms of the energy function \mathcal{E} defined in section 4.1. Recall the definition of the slice Σ_τ in section 2.3. In particular, vacuum data $(g, k) \in \Sigma_\tau$ satisfy the CMCSH gauge conditions.

Definition 5.1. *Let (g, k) be a vacuum data set on M with mean curvature τ . Let λ be given by (4.21) and let (\tilde{g}, \tilde{k}) be the rescaled metric and second fundamental form as defined in (4.20). Let $\mathcal{B}(\alpha)$ be the set of $(g, k) \in \Sigma_\tau^3$ so that*

$$\|\tilde{g} - \gamma\|_{H^3}^2 + \|\tilde{k} + \gamma\|_{H^2}^2 < \alpha.$$

We will say that (g, k) satisfies the smallness condition if $(g, k) \in \mathcal{B}(\alpha)$. \square

The smoothness of the scale-free Bel–Robinson energy \mathcal{E} , Lemma 4.2, together with the fact that the Hessian of \mathcal{E} , the scale-free Bel–Robinson energy restricted to the slice Σ_τ is positive definite with respect to the Sobolev norm $H^3 \times H^2$, Theorem 4.4, implies, by Taylor’s theorem, the following estimate.

Theorem 5.2. *Assume that M is rigid. There is an $\alpha > 0$ so that for $(g, k) \in \mathcal{B}(\alpha)$, there is a constant $D(\alpha) < \infty$, depending only on α and the topology of M , such that*

$$D(\alpha)^{-1} \mathcal{E}(g, k) \leq \|\tilde{g} - \gamma\|_{H^3}^2 + \|\tilde{k} + \gamma\|_{H^2}^2 \leq D(\alpha) \mathcal{E}(g, k). \quad (5.1)$$

In view of the analysis of the elliptic defining equations for N, X in [1], there is a neighborhood of $(\gamma, -\gamma)$ in Σ_{-3}^3 , such that \hat{N}, X are small in $W^{1,\infty}$ norm, defined by $\|f\|_{W^{1,\infty}} = \|f\|_{L^\infty} + \|Df\|_{L^\infty}$. It is straightforward to check

Corollary 5.3. *Let $\alpha > 0$ be such that the conclusion of Theorem 5.2 holds. There is a constant $\delta > 0$ so that for $(g, k) \in \mathcal{B}(\alpha)$ with $\mathcal{E}(g, k) < \delta$, it holds that*

$$\max(\Lambda, \|\hat{N}\|_{W^{1,\infty}}, \|X\|_{W^{1,\infty}}, \|g\|_{W^{1,\infty}}, \|k\|_{L^\infty}) < 1/\delta.$$

Lemma 5.4. *Let $\alpha > 0$ be small enough so that the conclusion of Theorem 5.2 holds. Let $(g, k) \in \mathcal{B}(\alpha)$ and let N, X be the corresponding solutions of the defining equations (2.8). Then there is a constant C such that*

$$\|\tilde{\hat{k}}\|_{L^\infty} \leq C\mathcal{E}^{\frac{1}{2}}, \quad (5.2a)$$

$$\|\tilde{\hat{N}}\|_{L^\infty} \leq C\mathcal{E}, \quad (5.2b)$$

$$\|\widetilde{\nabla N}\|_{L^\infty} \leq C\mathcal{E}, \quad (5.2c)$$

$$\|\tilde{\hat{\pi}}\|_{L^\infty} \leq C\mathcal{E}^{\frac{1}{2}}. \quad (5.2d)$$

Proof. The inequality (5.2a) follows from the definition of \mathcal{B} and Sobolev imbedding. The Lapse equation (2.8a) implies by the maximum principle,

$$|\hat{N}| \leq \frac{3}{\tau^2} \frac{\|\hat{k}\|_{L^\infty}^2}{\|\hat{k}\|_{L^\infty}^2},$$

which gives (5.2b) after rescaling. A standard elliptic estimate gives (5.2c). Finally, (4.7) together with the above estimates yield $\|\tilde{\hat{\pi}}\|_{L^\infty} \leq C(\mathcal{E}^{\frac{1}{2}} + \mathcal{E})$, which after using the smallness assumption and redefining C gives

$$\|\tilde{\hat{\pi}}\|_{L^\infty} \leq C\mathcal{E}^{\frac{1}{2}}.$$

□

5.1. Differential inequalities for the Rescaled Bel–Robinson energies. In this section we will estimate the time derivatives $\partial_\sigma \tilde{\mathcal{Q}}_i$ of the scale-free Bel–Robinson energies with respect to the logarithmic time σ defined in section 4.1.

Lemma 5.5. *Assume $(g, k) \in \mathcal{B}(\alpha)$ for α sufficiently small so that the conclusion of Theorem 5.2 holds. Then*

$$\partial_\sigma \tilde{\mathcal{Q}}_0(\sigma, W) \leq -(2 - 2C\mathcal{E}^{1/2})\tilde{\mathcal{Q}}_0, \quad (5.3)$$

Proof. Replacing all the fields in the RHS of (4.11) by their scale-free versions, noting in particular that $\tilde{\tau} = -3$ and $t < 0$, we get

$$\partial_\sigma \tilde{\mathcal{Q}}_0 = -2\tilde{\mathcal{Q}}_0 + \tilde{F}_1$$

where \tilde{F}_1 is the scalefree version of

$$F_1 = -9 \int_{M_\tau} N Q_{abcd} \hat{\pi}^{ab} T^c T^d d\mu_{M_\tau} + 3\tau \int_{M_\tau} \hat{N} Q_{TTTT}$$

The maximum principle applied to the Lapse equation (2.8a) implies, after a rescaling, that $\tilde{N} \leq \frac{1}{3}$. This gives the estimate

$$\tilde{F}_1 \leq C(\|\tilde{\pi}\|_{L^\infty} + \|\tilde{N}\|_{L^\infty}) \tilde{\mathcal{Q}}_0(\tau, W).$$

To finish the proof note that by Lemma 5.4,

$$\|\tilde{\pi}\|_{L^\infty} + \|\tilde{N}\|_{L^\infty} \leq C\mathcal{E}^{1/2},$$

using the smallness assumption. We write the resulting inequality in the form (5.3) for convenience. \square

Remark 5.1. *The proof of Lemma 5.5 gives the inequality*

$$\partial_\sigma \tilde{\mathcal{Q}}_0 \leq -(2 - C(\|\tilde{\pi}\|_{L^\infty} + \|\tilde{N}\|_{L^\infty})) \tilde{\mathcal{Q}}_0,$$

which is valid without the smallness assumption. \square

Lemma 5.6. *Assume $(g, k) \in \mathcal{B}(\alpha)$ for α sufficiently small so that the conclusion of Theorem 5.2 holds. Then*

$$\partial_\sigma \mathcal{E} \leq -(2 - 2C\mathcal{E}^{\frac{1}{2}})\mathcal{E}. \quad (5.4)$$

Proof. In view of the smallness condition, the inequality (5.2b) and Lemma 5.5, we only need to consider $\partial_\sigma \tilde{\mathcal{Q}}_1$. We proceed as in the proof of Lemma 5.5, using the scale-free version of (4.12) taking into account $\tilde{\tau} = -3$, we get

$$\partial_\sigma \tilde{\mathcal{Q}}_1 = -2\tilde{\mathcal{Q}}_1 + \tilde{F}_2$$

where \tilde{F}_2 is the scale-free version of

$$\begin{aligned} F_2 = & -6 \int_{M_\tau} N \mathcal{G}_1(W) d\mu_{M_\tau} - 9 \int_{M_\tau} N Q(\bar{\nabla}_T W)_{abcd} \hat{\pi}^{ab} T^c T^d d\mu_{M_\tau} \\ & + 5\tau \int_{M_\tau} \hat{N} Q(\bar{\nabla}_T W)_{TTTT} \end{aligned}$$

We see that in order to estimate \tilde{F}_2 , we need to estimate $\mathcal{G}_1(W)$, which is given in Lemma 4.1. Taking into account the detailed structure of \mathcal{G}_1 , cf. Remark 4.1 we get the estimate

$$\begin{aligned} \int N \mathcal{G}_1(W) \leq & C \left[\|\hat{\pi}\|_\infty (1 + \|k\|_\infty) (\mathcal{Q}_0 + \mathcal{Q}_1) \right. \\ & \left. + \int_{M_\tau} N |E(\bar{\nabla}_T W)| (|E(W)|^2 + |B(W)|^2) d\mu_{M_\tau} \right]. \end{aligned}$$

By the Holder inequality,

$$\int |E(\bar{\nabla}_T W)| (|E(W)|^2 + |B(W)|^2) \leq C \mathcal{Q}_1^{1/2} (\|E(W)\|_{L^4}^2 + \|B(W)\|_{L^4}^2).$$

By the Sobolev inequality, we may bound the scalefree version of

$$\|E(W)\|_{L^4}^2 + \|B(W)\|_{L^4}^2$$

by $C\mathcal{E}$. We now have the estimate

$$\int \tilde{N}\tilde{\mathcal{G}}_1 \leq C \left(\|\tilde{\pi}\|_\infty (1 + \|\tilde{k}\|_\infty) \mathcal{E} + \mathcal{E}^{3/2} \right) \quad (5.5)$$

$$\leq C\mathcal{E}^{3/2}, \quad (5.6)$$

for $(g, k) \in \mathcal{B}(\alpha)$. Proceeding similarly with the other terms in \tilde{F}_2 yields an inequality which we write in the form (5.4) for convenience. \square

6. GLOBAL EXISTENCE

Fix $\tau_0 < 0$ and let $(g(\tau_0), k(\tau_0))$ be data for Einstein's equations with mean curvature τ_0 and assume that $(g(\tau_0), k(\tau_0)) \in \mathcal{B}(\alpha)$ for an $\alpha > 0$ small enough so that the conclusion of Theorem 5.2 holds.

We have seen above that for small data, the second order scale-free Bel-Robinson energy \mathcal{E} satisfies the differential inequality (5.4). We will use this to prove

Theorem 6.1 (Global existence for small data). *Assume that M is rigid. Let $\alpha > 0$ be such that the conclusion of Theorem 5.2 holds. There is an $\epsilon \in (0, \alpha)$ small enough that if $(g^0, k^0) \in \mathcal{B}(\epsilon)$, then the maximal existence interval in mean curvature time τ , for the vacuum Einstein equations in CMCSH gauge, with data (g^0, k^0) is of the form $(T_-, 0)$. In particular, the CMCSH vacuum Einstein equations have global existence in the expanding direction for initial data in $\mathcal{B}(\epsilon)$.*

Here ϵ can be chosen as

$$\epsilon = D(\alpha)^{-1} \min(\delta, C^{-2}),$$

where $D(\alpha)$ is defined in Theorem 5.2, $\delta > 0$ is given by Corollary 5.3, and C is the constant in (5.4).

Proof. Under the assumptions of the theorem, by Theorem 5.2,

$$\mathcal{E}(g^0, k^0) < \min(\delta, C^{-2}) \quad (6.1)$$

holds. Thus the conclusion of Corollary 5.3 holds. By the definition of $\mathcal{B}(\alpha)$ we may apply Theorem 2.1 to conclude that there is a nontrivial maximal existence interval (T_-, T_+) in mean curvature time τ , with $T_+ \leq 0$, for (g^0, k^0) in $H^3 \times H^2$. We will assume $T_+ < 0$ and prove that this leads to a contradiction, using energy estimates and the continuation principle.

Let $y(\sigma)$ be the solution to the initial value problem

$$\frac{dy}{d\sigma} = -2y + 2Cy^{3/2}, \quad y(\sigma_0) = y_0. \quad (6.2)$$

Then if $y_0 = \mathcal{E}(g^0, k^0)$, is such that $y(\sigma) < \infty$ for $\sigma \in [\sigma_0, \sigma_+]$, we have

$$\mathcal{E}(\sigma) \leq y(\sigma), \quad \sigma \in [\sigma_0, \sigma_+).$$

The solution to (6.2) is

$$y^{-1/2} = C + e^{\sigma - \sigma_0}(y_0^{-1/2} - C),$$

if $y_0 < C^{-2}$, and in this case $y(\sigma) < y(\sigma_0)$ for $\sigma \in (\sigma_0, \infty)$. This means that if (6.1) holds at $\sigma = \sigma_0$, it holds for $\sigma \in [\sigma_0, \sigma_+)$. By Theorem 5.2, this implies that $\|\tilde{g} - \gamma\|_{H^3} + \|\tilde{k} + \gamma\|_{H^2}$ is uniformly bounded for $\sigma \in [\sigma_0, \sigma_+)$. By Corollary 5.3, this implies that the inequality

$$\sup_{\sigma \in [\sigma_0, \sigma_+)} (\Lambda[\tilde{g}] + \|D\tilde{g}\|_{L^\infty} + \|k\|_{L^\infty}) < \delta^{-1},$$

holds.

In view of the continuation principle, Point 2 of Theorem 2.1, this contradicts the assumption that (T_-, T_+) is the maximal existence interval in mean curvature time τ , with $T_+ < 0$. It follows that $T_+ = 0$ which completes the proof. \square

6.1. Geodesic completeness.

Theorem 6.2. *Let (M, g^0, k^0) and (\bar{M}, \bar{g}) be as in Theorem 6.1. Then (\bar{M}, \bar{g}) is causally geodesically complete in the expanding direction.*

Proof. By Theorem 6.1, (\bar{M}, \bar{g}) is globally foliated by CMC hypersurfaces to the future of (M, g^0, k^0) , i.e. in the expanding direction, with $t = \text{tr}k \nearrow t_* = 0$.

Let $c(\lambda)$ be a future directed causal geodesic, with affine parameter λ . Let

$$u = \frac{dc}{d\lambda}, \quad \langle u, u \rangle = \begin{cases} -1 \\ 0 \end{cases},$$

be the normalized velocity, where $\langle \cdot, \cdot \rangle = \bar{g}(\cdot, \cdot)$. The geodesic equation is

$$\bar{\nabla}_u u = 0. \tag{6.3}$$

As c is causal, we may use t as parameter. Let

$$u^0 = dt(u) = \frac{dt}{d\lambda}.$$

In order to prove geodesic completeness in the expanding direction, it is sufficient to prove that the solution to the geodesic equation exists for an infinite interval of the affine parameter, i.e. $\lim_{t \nearrow t_*} \lambda(t) = \infty$ or

$$\lim_{t \nearrow t_*} \int_{t_0}^t \frac{d\lambda}{dt} dt = \infty,$$

or using the definition of u^0 ,

$$\lim_{t \nearrow t_*} \int_{t_0}^t \frac{1}{u^0} dt = \infty. \tag{6.4}$$

Suppose that we are able to prove that Nu^0 is bounded from above as $t \nearrow t_*$. Then (6.4) holds precisely when

$$\lim_{t \nearrow t_*} \int_{t_0}^t N dt = \infty.$$

For a function f on \bar{M} , we have

$$\frac{df(c(t))}{dt} = \frac{d\lambda}{dt} \frac{df(c(t))}{d\lambda} = \left(\frac{dt}{d\lambda} \right)^{-1} \frac{df(c(t))}{d\lambda} = \frac{1}{u^0} \bar{\nabla}_u f. \quad (6.5)$$

A calculation in local coordinates using the 3+1 form of \bar{g} , gives $T_\mu = -N\delta_\mu^0$ where δ_μ^ν is the Kronecker delta, i.e. $\langle T, V \rangle = -Ndt(V)$ for any V . This shows that

$$-Nu^0 = \langle u, T \rangle,$$

or

$$u = Nu^0 T + Y, \quad (6.6)$$

where Y is tangent to M . Let $\epsilon = 0, 1$. Then by assumption, $\langle u, u \rangle = -|\epsilon|$ which using (6.6) gives

$$|Y|_g^2 = N^2(u^0)^2 - |\epsilon|.$$

In particular, we get the inequality

$$|Y|_g \leq Nu^0. \quad (6.7)$$

A computation in a Fermi propagated frame gives using (6.6),

$$\bar{\nabla}_u T = Nu^0 N^{-1} \nabla_i N e_i - k_{ij} Y^j e_i. \quad (6.8)$$

By our choice of time orientation we have $N > 0$ and $u^0 > 0$ and $\text{tr} k < 0$.

We now compute using $\bar{\nabla}_u u = 0$, (6.6) and (6.8)

$$\begin{aligned} \frac{d}{dt} \ln(Nu^0) &= -\frac{1}{Nu^0} \frac{d}{dt} \langle u, T \rangle \\ &= -\frac{1}{N(u^0)^2} \langle u, \bar{\nabla}_u T \rangle \\ &= -\frac{1}{N(u^0)^2} (u^0 \nabla_Y N - k_{ij} Y^i Y^j) \\ &= -\frac{\nabla_Y N}{Nu^0} + N \hat{k}_{ij} \frac{Y^i}{Nu^0} \frac{Y^j}{Nu^0} + N \frac{\text{tr} k}{3} \frac{|Y|_g^2}{N^2(u^0)^2} \end{aligned}$$

use (6.7) and $\text{tr} k < 0$,

$$\leq \|\nabla N\|_{L^\infty;g} + \|N\hat{k}\|_{L^\infty;g}$$

use scaling properties of N, \hat{k}, g , cf. section 4.1,

$$\leq \|\widetilde{\nabla N}\|_{L^\infty;\tilde{g}} + \lambda^{-1} \|\tilde{N}\tilde{\hat{k}}\|_{L^\infty;\tilde{g}},$$

with $\lambda = \text{tr}k/3 = t/3$. By the proof of the global existence result Theorem 6.1, we have $\mathcal{E}(t) \leq Ct^2$ and \tilde{g} is close to γ . Therefore by Sobolev imbedding, we can relate the norms w.r.t. \tilde{g} to the norms w.r.t. γ and we get

$$\frac{d}{dt} \ln(Nu^0) \leq C \left(\|\widetilde{\nabla N}\|_{L^\infty} + \lambda^{-1} \|\tilde{N}\tilde{k}\|_{L^\infty} \right).$$

Now an application of the estimate (5.2) together with the decay of \mathcal{E} gives

$$\begin{aligned} \|\widetilde{\nabla N}\|_{L^\infty} &\leq Ct^2, \\ \|\tilde{N}\tilde{k}\|_{L^\infty} &\leq Ct, \end{aligned}$$

which in view of $\lambda^{-1}t = 3$ gives

$$\ln(Nu^0)(t) - \ln(Nu^0)(t_0) \leq C,$$

and hence $\ln(Nu^0) \leq C$ for some constant C as $t \nearrow t_* = 0$.

We have now proved that Nu^0 is bounded from above and therefore it is sufficient to prove that

$$\lim_{t \nearrow t_*} \int_{t_0}^t N dt = \infty. \quad (6.9)$$

Write $N = \hat{N} + \frac{3}{t^2}$ as in (4.10). Using (5.2) and the scaling rule (4.20) to estimate \hat{N} gives $N \geq C/t^2$ as $t \nearrow t_* = 0$. This shows that (6.9) holds and completes the proof of Lemma 6.2. \square

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APPENDIX A. BASIC DEFINITIONS AND IDENTITIES

A.1. Conventions. We begin by recalling some basic facts and definitions. We use the following conventions for curvature.

The Riemann tensor is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

In a coordinate frame $\{e_a\}$ we have

$$R^d_{cab} Z^c = \nabla_a \nabla_b Z^d - \nabla_b \nabla_a Z^d.$$

This gives the conventions for index calculations

$$[\nabla_a, \nabla_b] t_c = R_{abc}{}^d t_d, \quad (A.1)$$

$$[\nabla_a, \nabla_b] t^c = R^c_{dab} t^d. \quad (A.2)$$

The Ricci curvature and the scalar curvature are defined (in an ON frame) by

$$\text{Ric}(X, Y) = \sum_i \langle R(e_i, X)Y, e_i \rangle, \quad (\text{A.3})$$

$$\text{Scal} = \sum_i \text{Ric}(e_i, e_i), \quad (\text{A.4})$$

or in index notation

$$R_{ij} = g^{kl} R_{ikjl}, \quad R = g^{ij} R_{ij}.$$

Note also

$$\text{Ric}(X, Y) = \text{tr}(Z \mapsto R(Z, X)Y).$$

The Riemann tensor satisfies the Bianchi identities

$$\nabla_{[e} R_{ab]cd} = \frac{1}{3} (\nabla_e R_{abcd} + \nabla_a R_{becd} + \nabla_b R_{eacd}) = 0.$$

The trace free part of the Riemann tensor in an n -dimensional manifold is

$$\begin{aligned} C_{abcd} &= R_{abcd} - \frac{1}{n-2} (g_{ac} R_{bd} + g_{bd} R_{ac} - g_{bc} R_{ad} - g_{ad} R_{bc}) \\ &\quad + \frac{1}{(n-1)(n-2)} (g_{ac} g_{bd} - g_{ad} g_{bc}) R. \end{aligned} \quad (\text{A.5})$$

The totally anti-symmetric tensor ϵ in dimension $3+1$ satisfies the identities

$$\begin{aligned} \epsilon^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \epsilon_{\beta_1 \beta_2 \beta_3 \beta_4} &= -\det(\delta_{\beta_j}^{\alpha_i})_{i,j=1,\dots,4}, \\ \epsilon^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \epsilon_{\alpha_1 \beta_2 \beta_3 \beta_4} &= -\det(\delta_{\beta_j}^{\alpha_i})_{i,j=2,\dots,4}, \\ \epsilon^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \epsilon_{\alpha_1 \alpha_2 \beta_3 \beta_4} &= -2 \det(\delta_{\beta_j}^{\alpha_i})_{i,j=3,\dots,4}, \\ \epsilon^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \epsilon_{\alpha_1 \alpha_2 \alpha_3 \beta_4} &= -6 \delta_{\beta_4}^{\alpha_4}, \\ \epsilon^{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \epsilon_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} &= -24. \end{aligned} \quad (\text{A.6a})$$

In an ON frame adapted to a spacelike hypersurface M in a $3+1$ dimensional manifold we define (c.f. [5, p. 144])

$$\epsilon_{ijk} = \epsilon_{Tijk}. \quad (\text{A.7})$$

$$\epsilon^{i_1 i_2 i_3} \epsilon_{j_1 j_2 j_3} = \det(\delta_{j_l}^{i_k})_{k,l=1,2,3} = 6 \delta_{j_1}^{[i_1} \delta_{j_2}^{i_2} \delta_{j_3}^{i_3]}, \quad (\text{A.8a})$$

$$\epsilon^{i_1 i_2 i_3} \epsilon_{i_1 j_2 j_3} = \det(\delta_{j_l}^{i_k})_{k,l=2,3} = 2 \delta_{j_2}^{[i_2} \delta_{j_3}^{i_3]}, \quad (\text{A.8b})$$

$$\epsilon^{i_1 i_2 i_3} \epsilon_{i_1 i_2 j_3} = 2 \delta_{j_3}^{i_3}, \quad (\text{A.8c})$$

$$\epsilon^{i_1 i_2 i_3} \epsilon_{i_1 i_2 i_3} = 6. \quad (\text{A.8d})$$

In dimension 3 we have the duality relations

$$\xi_{ab} = \epsilon_{ab}^m \eta_m,$$

for $\xi_{ab} = \xi_{[ab]}$, where

$$\eta_m = \frac{1}{2} \epsilon_m^{ab} \xi_{ab}.$$

A.2. Operations on symmetric 2-tensors. Define the following operations on symmetric 2-tensors on a 3-dimensional Riemann manifold:

$$A \cdot B = A_{ab} B^{ab}, \quad (\text{A.9})$$

$$(A \wedge B)_a = \epsilon_a^{bc} A_b^d B_{dc}, \quad (\text{A.10})$$

$$(v \wedge A)_{ab} = \epsilon_a^{cd} v_c A_{db} + \epsilon_b^{cd} v_c A_{ad}, \quad (\text{A.11})$$

$$(A \times B)_{ab} = \epsilon_a^{cd} \epsilon_b^{ef} A_{ce} B_{df} + \frac{1}{3} (A \cdot B) g_{ab} - \frac{1}{3} (\text{tr} A)(\text{tr} B) g_{ab}, \quad (\text{A.12})$$

$$\text{curl} A_{ab} = \frac{1}{2} (\epsilon_a^{cd} \nabla_d A_{cb} + \epsilon_b^{cd} \nabla_d A_{ca}), \quad (\text{A.13})$$

$$\text{div} A_a = \nabla^b A_{ab}. \quad (\text{A.14})$$

The operation \wedge is skew symmetric, while \times is symmetric, and the identities

$$A \cdot (v \wedge B) = -2v \cdot (A \wedge B)$$

$$A \cdot (B \times C) = (A \times B) \cdot C \quad (\text{if } \text{tr} A = \text{tr} C = 0)$$

hold. The expression $A \times B$ can be expanded as

$$\begin{aligned} (A \times B)_{ab} &= A_a^c B_{cb} + A_b^c B_{ca} \\ &\quad - \frac{2}{3} (A \cdot B) g_{ab} + \frac{2}{3} (\text{tr} A)(\text{tr} B) g_{ab} - (\text{tr} A) B_{ab} - (\text{tr} B) A_{ab}. \end{aligned}$$

A computation shows

$$\text{div}(A \wedge B) = -(\text{curl} A) \cdot B + A \cdot (\text{curl} B). \quad (\text{A.15})$$

Let A be a symmetric covariant 2-tensor on \bar{M} and suppose A is t -tangent, i.e. $A_{\alpha\beta} T^\beta = 0$. Then in a Fermi propagated frame,

$$\bar{\nabla}_T A_{ij} = T A_{ij} \quad (\text{A.16})$$

$$\begin{aligned} \mathcal{L}_{\partial_t} A_{ij} &= N \bar{\nabla}_T A_{ij} - N \left((k \times A)_{ij} + \frac{2}{3} (k \cdot A) g_{ij} \right. \\ &\quad \left. - \frac{2}{3} (\text{tr} k)(\text{tr} A) g_{ij} + (\text{tr} A) k_{ij} + (\text{tr} k) A_{ij} \right). \end{aligned} \quad (\text{A.17})$$

Define the covariant exterior derivative $d^\nabla u$ on symmetric 2-tensors by

$$(d^\nabla u)_{ijk} = \nabla_k u_{ij} - \nabla_j u_{ik}. \quad (\text{A.18})$$

The operators $\text{curl}, \text{div}, d^\nabla$ are related by [5, p. 103]

$$d^\nabla u_{kij} = \left(\text{curl} u_{kl} + \frac{1}{2} (\text{div} u_m - \nabla_m \text{tr} u) \epsilon^m_{kl} \right) \epsilon^l_{ij}. \quad (\text{A.19})$$

Taking into account the symmetry of curl this implies

$$|d^\nabla u|^2 = 2(|\text{curl} u|^2 + \frac{1}{2} |\text{div} u - \nabla \text{tr} u|^2). \quad (\text{A.20})$$

If u has compact support then in dimension n ,

$$\int_M |d^\nabla u|^2 = 2 \int_M |\nabla u|^2 - 2 \int_M |\operatorname{div} u|^2 + 2 \int_M (u_k^l R_{ijl}^k + u_{ik} R_j^k) u^{ij}. \quad (\text{A.21})$$

This leads to, if $\operatorname{tr} u = 0$,

$$\int_M (|\nabla u|^2 + 3R_{ij} u^{ki} u_k^j - \frac{1}{2} R |u|^2) = \int_M (|\operatorname{curl} u|^2 + \frac{3}{2} |\operatorname{div} u|^2), \quad (\text{A.22})$$

in case M is of dimension 3. If we further restrict to (M, γ) with γ hyperbolic, so that $R[\gamma] = -6$, we get

$$\int_M (|\operatorname{curl} u|^2 + \frac{3}{2} |\operatorname{div} u|^2) = \int_M (|\nabla u|^2 - 3|u|^2). \quad (\text{A.23})$$

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